Resurgence, analytic continuation and toward a continuum definition of QFT

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In collaboration with
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Philip Argyres (4d QFT, QCD(adj))

+ Gokce Basar, Aleksey Cherman, Daniele Dorigoni, ...

Motivation: Can we make sense out of QFT? When is there a continuum definition of QFT?

Dyson(50s), 't Hooft (77),

Quoting from M. Douglas comments, in Foundations of QFT, talk at String-Math 2011

"A good deal of mathematical work starts with the Euclidean functional integral (as we will). There is no essential difficulty in rigorously defining a Gaussian functional integral, in setting up perturbation theory, and in developing the BRST and BV formulations (see e.g. K. Costello's work).

A major difficulty, indeed many mathematicians would say the main reason that QFT is still "not rigorous," is that standard perturbation theory only provides an asymptotic (divergent) expansion. There is a good reason for this, namely exact QFT results are not (often) analytic in a finite neighborhood of zero coupling.

Recently, few people are attempting to answer this question, whether/when a n.p. continuum definition of QFT may exist and reinvigorate this problem.

Argyres, Dunne, MÜ: Resurgence in QFTs, QM, and path integrals
Schiappa, Marino,..:Resurgence in string theory and matrix models
Kontsevich: recent talk at PI, Resurgence from the path integral perspective
Garoufalidis, Costin: Math and Topological QFTs

The common concept, which all these folks seem to be highly influenced by (and which is virtually unknown in physics community) is a "recent" mathematical progress, called

Resurgence Theory, developed by Jean Ecalle (80s)

and applied to QM by Pham, Delabaere, Voros.
(also relevant Dingle-Berry-Howls)

Ecalle's theory changed (will change?) the overall perspective on asymptotic analysis, for both mathematicians and physicists alike.

Earlier hints that a resurgent structure must underlie QFT.

CP(N-1) model on R2 and standard problems verbatim in 4d QCD.

An asymptotically free non-linear sigma model with a complex projective target space. Large-N, successful. Many problems are still unresolved at finite-N.

1) Perturbation theory is an asymptotic *(divergent) expansion even after regularization and renormalization*. Is there a meaning to perturbation theory?

2) Invalidity of the semi-classical dilute instanton gas approximation on R2. (e.g,Affleck). DIG assumes inter-instanton separation is much larger than the instanton size, but the latter is a moduli, hence no meaning to the assumption. (Clever work of Fateev, Frolov, Schwarz (79) does not address this issue.)

3) ``Infrared embarrassment",e.g., large-instanton contribution to vacuum energy is IR-divergent, see Coleman's lectures.

4) A resolution of 2) was put forward by considering the theory in a small thermal box. But in the weak coupling regime, the theory always lands on the deconfined "regime". (Affleck, 80) So, *no semi-classical approximation for the confined regime* to date is found, (except a supersymmetric version of the theory, due to reasons not necessarily related to supersymmetry).

5) Incompatibility of large-N results with instantons. Obvious. (Witten, Jevicki, 79)

6) The renormalon ambiguity (technical, but deeper, to be explained), ('t Hooft,79).

CP(N-1) model on R2 and standard problems

No exaggeration in saying that what "our inheritance" from few generation earlier is a disaster. To be fair, they at least stated the problems of a typical QFT.

After mid-80's, very few serious field theorists worked on these type of problems, primarily due to sociological reasons, not because the problems were uninteresting, or progress was impossible.

If we are going to make progress in some foundational aspects of QFT, it is, of course, preferable to have a formalism of *practical utility,* whose results can be compared with *numerical experiments,* i.e., lattice field theory (a black box).

So far, there had been no such useful continuum formulation of general QFTs. "Constructive QFT" (Glimm, Jaffe, Spencer, Bridges) only attempts the first problem, but no success. (It was also detached from physics, in my opinion.)

This talk: Report progress in this direction, and argue a useful n.p. definition may underly the resolution of *all* puzzles/problems quoted above.

Simpler question: Can we make sense of the semi-classical expansion of QFT?

Argyres, MÜ,
Dunne, MÜ, 2012

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)}\,(\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n}\, e^{-n\,A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)}\,(\lambda\hbar)^k$$

pert. th. n-instanton factor pert. th. around n-instanton

All series appearing above are asymptotic, i.e., divergent as $c_{(0,k)} \sim k!$. The combined object is called trans-series following resurgence literature

Borel resummation idea: If $P(\lambda) \equiv P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$ has convergent Borel transform

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$

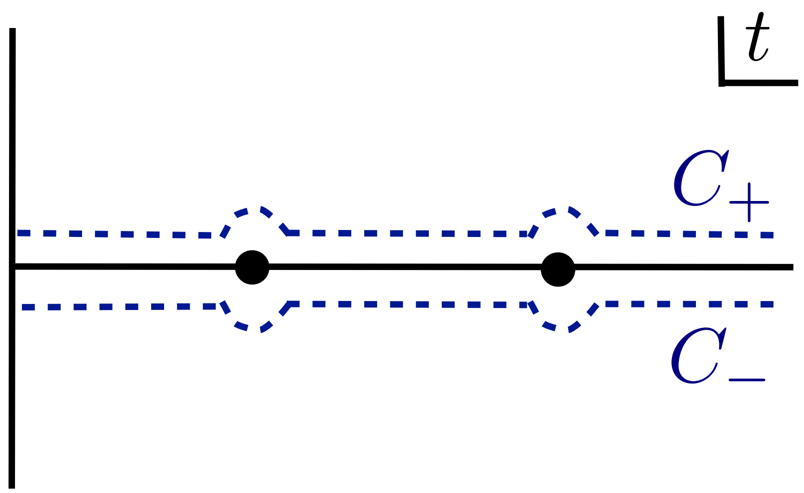
in neighborhood of $t = 0$, then

$$\mathbb{B}(g^2) = \frac{1}{g^2} \int_0^{\infty} BP(t) e^{-t/g^2} dt\ .$$

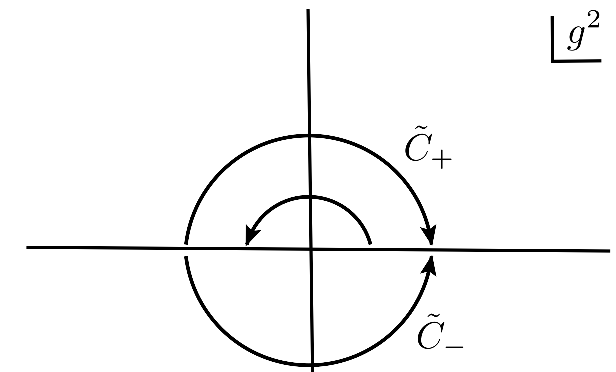
formally gives back $P(g^2)$, but is ambiguous if $BP(t)$ has singularities at $t \in \mathbb{R}^+$:

Borel plane and lateral (left/right) Borel sums

Directional (sectorial) Borel sum. $\mathcal{S}_\theta P(g^2) \equiv \mathbb{B}_\theta(g^2) = \frac{1}{g^2}\int_0^{\infty \cdot e^{i\theta}} BP(t)\, e^{-t/g^2}\, dt$



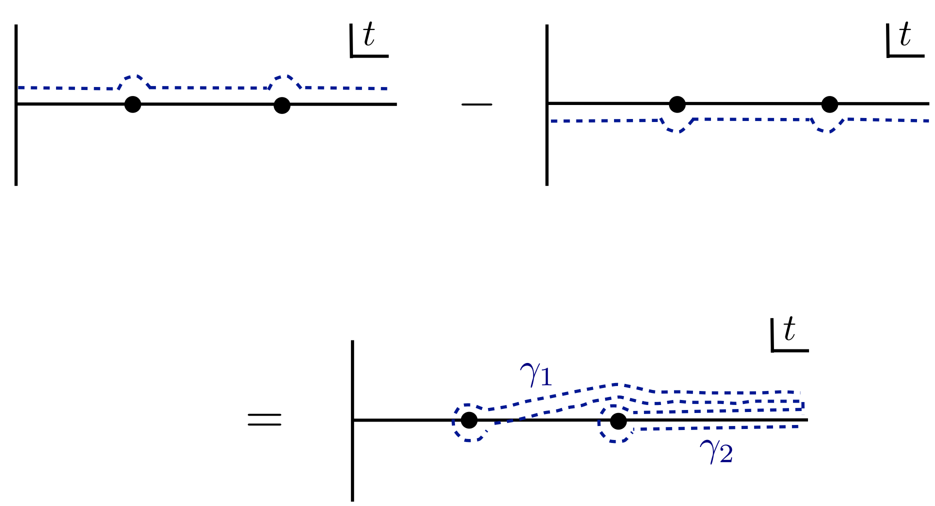
left and right Borel sums



Same structure by analytic continuation

$$\mathbb{B}_{0^\pm}(|g^2|) = \mathrm{Re}\,\mathbb{B}_0(|g^2|) \pm i\,\mathrm{Im}\,\mathbb{B}_0(|g^2|), \qquad \mathrm{Im}\,\mathbb{B}_0(|g^2|) \sim e^{-2S_I} \sim e^{-2A/g^2}$$

The *non-equality* of the left and right Borel sum means the series is *non-Borel summable or ambiguous*. The ambiguity has the same form of a 2-instanton factor **(not 1, never I).** The measure of ambiguity (Stokes automorphism/jump in g-space interpretation):



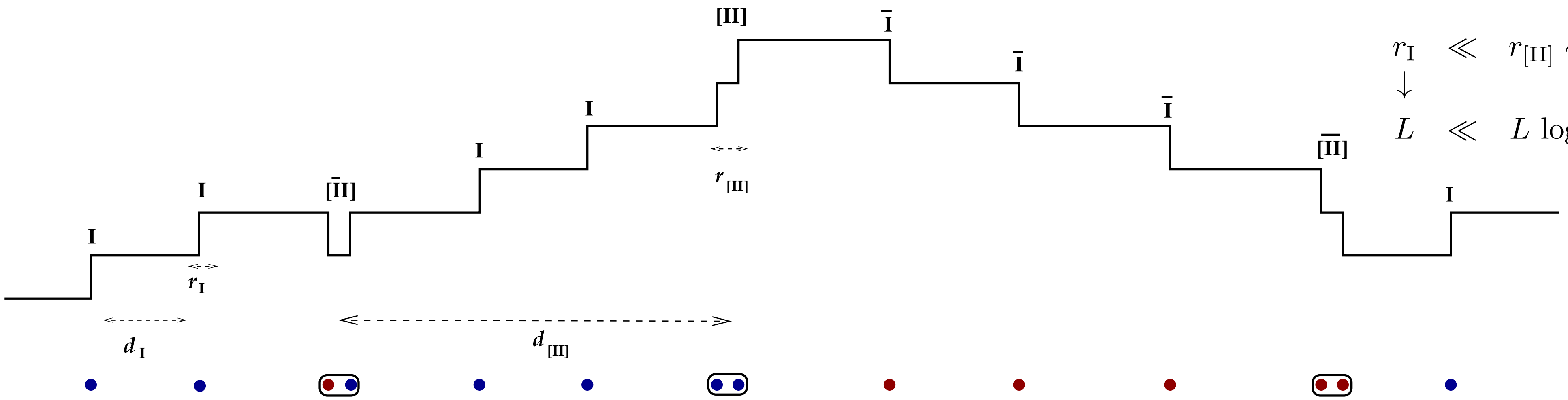
$$\mathcal{S}_{\theta^+} = \mathcal{S}_{\theta^-} \circ \underline{\mathfrak{S}}_\theta \equiv \mathcal{S}_{\theta^-} \circ \left(\mathbf{1} - \mathrm{Disc}_{\theta^-}\right),$$

$$\mathrm{Disc}_{\theta^-}\,\mathbb{B} \sim e^{-t_1/g^2} + e^{-t_2/g^2} + \ldots \qquad t_i \in e^{i\theta}\mathbb{R}^+$$

Jean Ecalle, 80s

Bogomolny--Zinn-Justin (BZJ) prescription

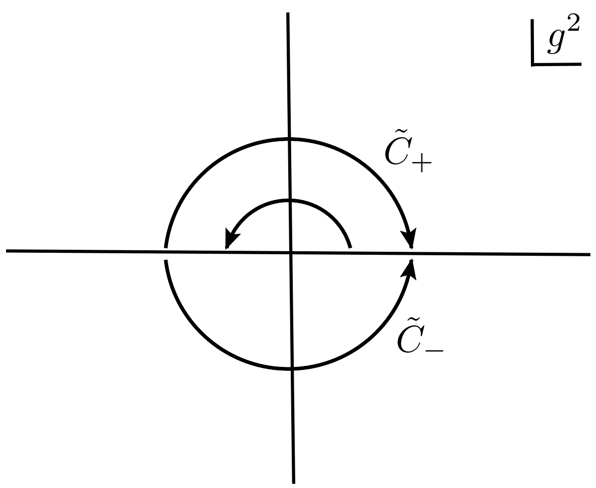
Bogomolny-Zinn-Justin prescription in QM (80s): done for double well potential, but consider a periodic potential. Dilute instanton, molecular instanton gas.



$$
\begin{array}{ccccccc}
r_{\rm I} & \ll & r_{[\rm II]} \sim \ell_{\rm qzm} & \ll & d_{\rm I} & \ll & d_{[\rm II]}, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \ll & L \log\left(\frac{1}{g^2}\right) & \ll & L\, e^{S_0} & \ll & L\, e^{2S_0}.
\end{array}
$$

How to make sense of topological molecules (or molecular instantons)? Why do we even need a molecular instanton? (Balitsky-Yung in SUSY QM, (86))

Naive calculation of I-anti-I amplitude: meaningless (why?) at $g^2 > 0$. The quasi-zero mode integral is dominated at small-separations where a molecular instanton is meaningless. Continue to $g^2 < 0$, evaluate the integral, and continue back to $g^2 > 0$: two fold-ambiguous!



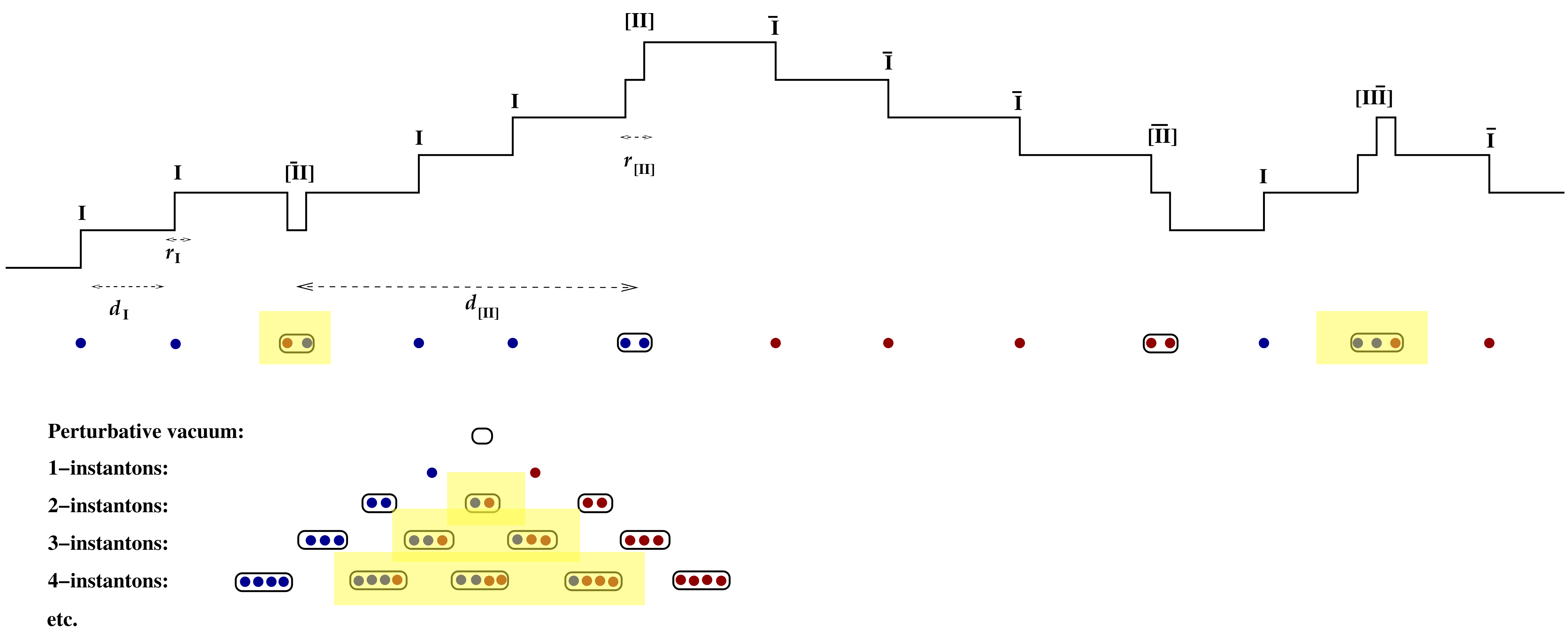
$$[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = \mathrm{Re}\,[\mathcal{I}\overline{\mathcal{I}}] + i\,\mathrm{Im}\,[\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm}$$

Because we are on Stokes line, later....

Remarkable fact: Leading ambiguities cancel. "N.P. CONFLUENCE EQUATION", elementary incidence of Borel-Ecalle summability which I will return:

$$\mathrm{Im}\, \mathbb{B}_{0,\theta=0^\pm} + \mathrm{Im}\, [\mathcal{I}\overline{\mathcal{I}}]_{\theta=0^\pm} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

The ambiguous topological configurations. All are non-BPS quasi-solutions!



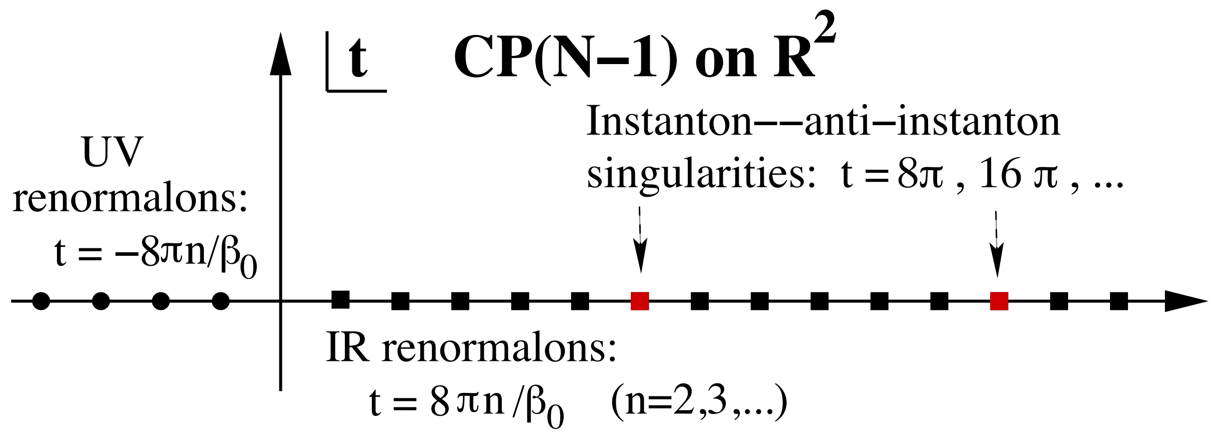
Can this work in QFT? QCD on R4 or CP(N-1) on R2?

't Hooft(79) :**No,** *on R4,* Argyres, MÜ: ***Yes,*** *on R3 x S1,*
F. David(84), Beneke(93) : **No,** *on R2.* Dunne, MÜ: ***Yes,*** *on R1 x S1*

Why doesn't it work, say for CP(N-1) on R2?

Instanton-anti-instanton contribution, calculated in some way, gives an $\pm i \exp[-2S_I]$.
Lipatov(77): Borel-transform BP(t) has singularities at $t_n = 2n\, g^2 S_I$. (Modulo the standard IR problems with 2d instantons, also see Bogomolny-Fateyev(77)).

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams!)



't Hooft called these IR-renormalon singularituies with the hope/expectation that they would be associated with a saddle point like instantons.
No such configuration is known!!

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from micro-dynamics?

Can this work in QFT? QCD on R4 or CP(N-1) on R2?

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Standard view in late 70s, from Parisi(78).

If the theory is renormalizable, the Borel transform has new singularities which cannot be controlled by using semi-classical methods [5-8] .

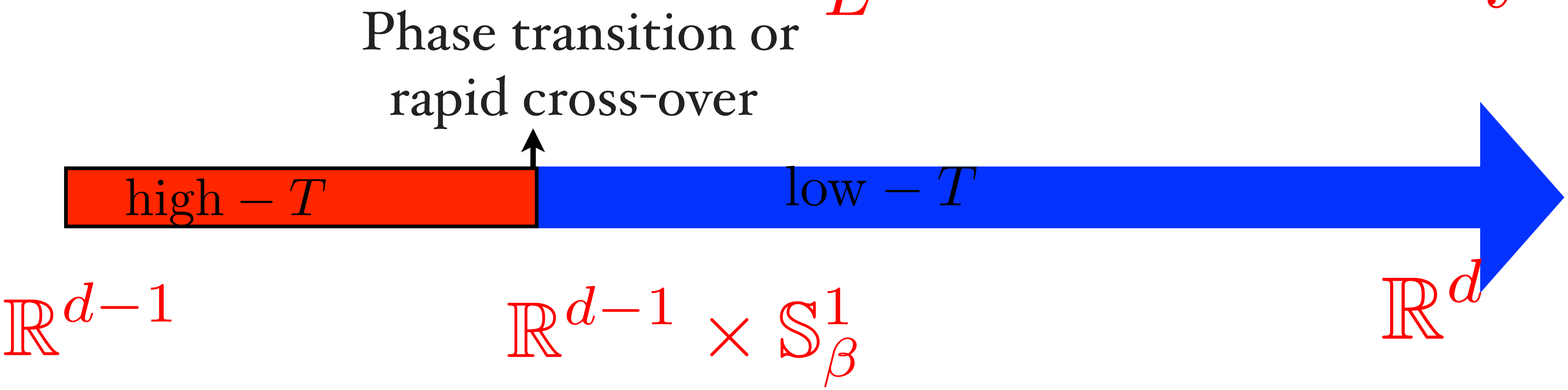
5 G. 't Hooft, Lectures given at Erice (1977)

6 B. Lautrup, Phys. Lett. 69B (1977) 109

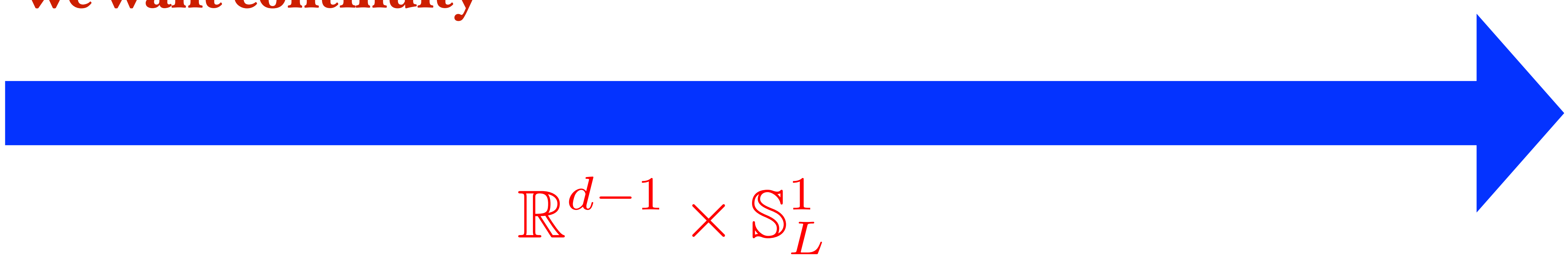
7 G. Parisi, Lectures given at the 1977 Cargèse Summer School

8 P. Olesen, Nordita preprint NBI HE 77.48 (1977)

$\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1_L$ and Continuity



We want continuity



Thermal finite-N: Rapid crossover at strong scale
Thermal large-N: Sharp phase transition at strong scale

Prevent both by using circle compactification or deformation.

$\mathbb{CP}^{N-1}$ on $\mathbb{R}^1 \times \mathbb{S}^1_L$ and Continuity

Thermal compactification is literally used thousands of times in field theories. For example, Affleck studied the theory on R1 x S1 to tame the instanton size moduli in the small-S1 regime. However, his study is only relevant for the deconfined regime of field theory, and is irrelevant to study the confined regime.

If we want to learn something pertinent to field theory on R2, we have to find a regime of the theory which is weakly coupled and continuously connected to the desired target theory. We call this, "principle of continuity".

Continuity is used in supersymmetry many times, starting with the supersymmetric (Witten) index calculation in early 80s.

In non-supersymmetric theories, the utility of continuity is realized in 2007 (Ünsal).
This point of view turns out to be truely useful (and I am fond of it).
I have explored many remarkable consequences of this simple idea with few collaborators, mainly, Poppitz, Yaffe, Shifman, Argyres, Dunne, Schäfer. I will discuss this idea for CP(N-1) here (with Dunne), along with the resurgence theory to provide an answer to the problems I mentioned earlier.

Sigma-connection holonomy (a new line operator)

Point-wise modulus and phase splitting, derivative of *each* phase transform as "gauge" connection.

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_N \end{pmatrix} = \begin{pmatrix} e^{i\varphi_1} \cos\frac{\theta_1}{2} \\ e^{i\varphi_2} \sin\frac{\theta_1}{2}\, \cos\frac{\theta_2}{2} \\ e^{i\varphi_3} \sin\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2}\, \cos\frac{\theta_3}{2} \\ \vdots \\ e^{i\varphi_N} \sin\frac{\theta_1}{2}\, \sin\frac{\theta_2}{2}\, \sin\frac{\theta_3}{2}\, \ldots \sin\frac{\theta_{N-1}}{2} \end{pmatrix}$$

$$\theta_i \in [0,\pi], \qquad \varphi_i \in [0,2\pi).$$

Build a new line operator, counter-part of the Wilson line, the sigma holonomy:

$$({}^L\Omega)_j(x_1) = \exp\left[i \int_0^L dx_2\, \mathcal{A}_{2,j} \right] = \exp\left[i(\varphi_j(x_1,0) - \varphi_j(x_1,L)) \right]$$

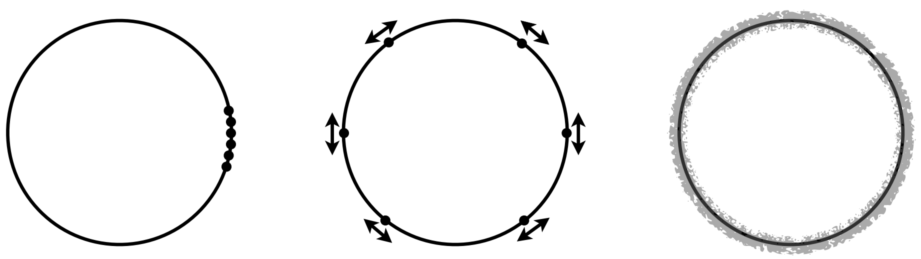
$${}^L\Omega(x_1) = \begin{pmatrix} e^{i[\varphi_1(x_1,0)-\varphi_1(x_1,L)]} & 0 & \ldots & 0 \\ 0 & e^{i[\varphi_2(x_1,0)-\varphi_2(x_1,L)]} & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & e^{i[\varphi_N(x_1,0)-\varphi_N(x_1,L)]} \end{pmatrix}$$

One-loop potential for Sigma holonomy

$$V_-[{}^L\Omega] = \frac{2}{\pi\beta^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(-1+(-1)^n N_f)(|\mathrm{tr}\,{}^L\Omega^n|-1) \qquad \text{(thermal)}$$

$$V_+[{}^L\Omega] = (N_f-1)\frac{2}{\pi L^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(|\mathrm{tr}\,{}^L\Omega^n|-1) \qquad \text{(spatial)}$$

Three types of holonomy

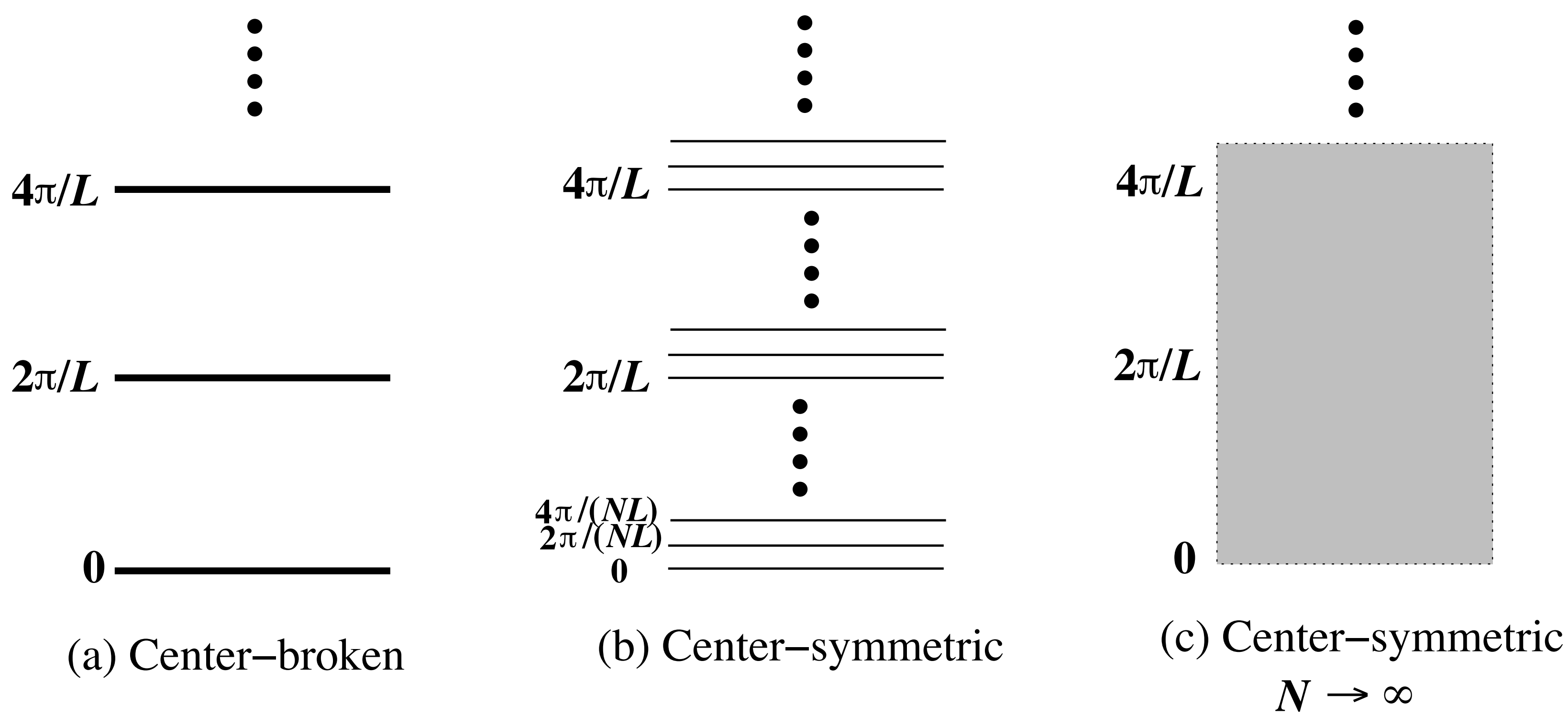


(a) Thermal (b) Spatial (c) Strong-coupling non-trivial hol.

Crucial difference of (a) and (b): **van Baal et. al.** in gauge theory on $R_3 \times S_1$

To achieve (b) in the $n_f = 0$ case require a deformation of the action. (b) is weak coupling realization of the center-symmetric background.

The dependence of perturbative spectrum to the sigma holonomy background



Same as gauge theory on R3 x S1, the fact that spectrum become dense in the last case is an imprint of the large-N volume independence (Eguchi-Kawai reduction). This is surprising on its own in a "vector"-model. I will not talk about it here.

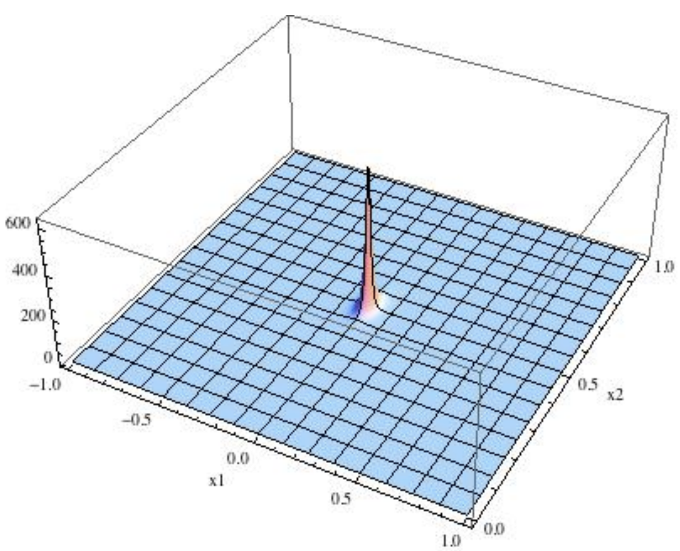
Instead, we will study non-pert. effects in the long-distance effective theory within Born-Oppenheimer approx. in case (b) for finite-N.

Topological configurations, 1-defects

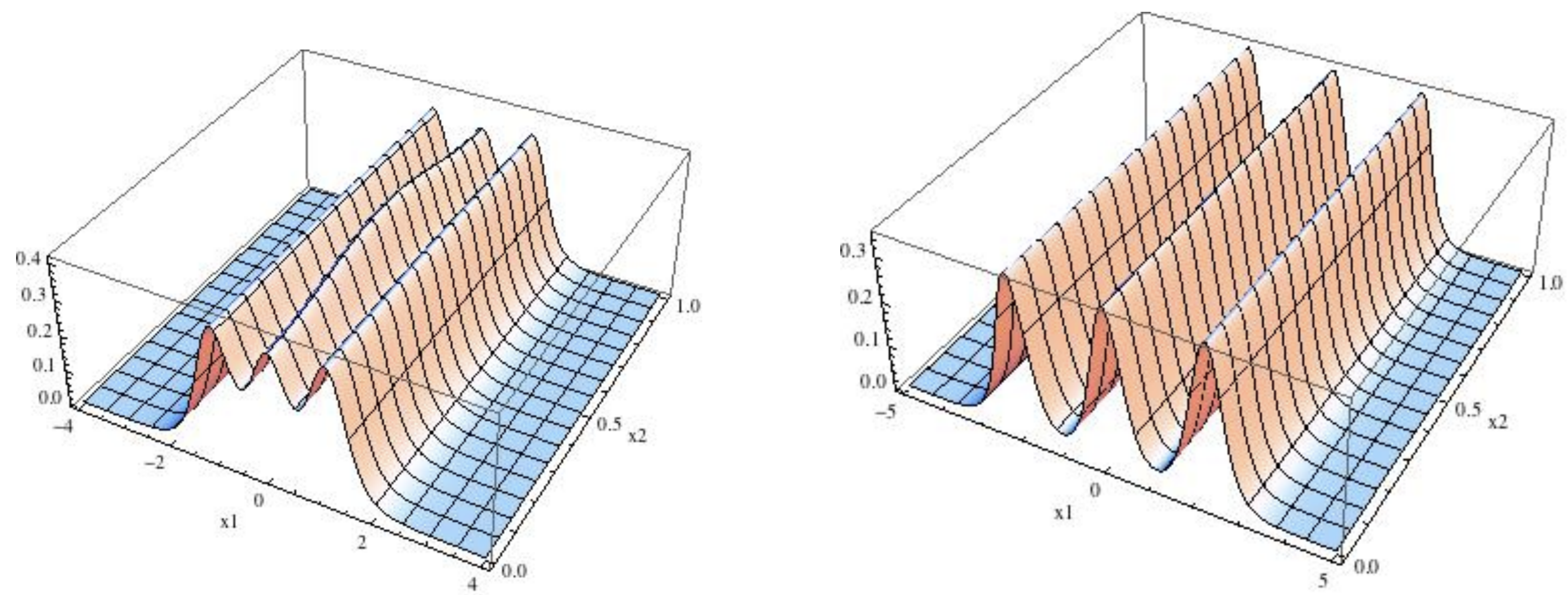
1-defects, Kink-instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra. The Nth type corresponds to the affine root and is present only because the theory is *locally 2d*! Also see Bruckmann (07), Brendel et.al.(09)

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i, \qquad \alpha_i \in \Gamma_r^\vee$$

$$\mathcal{K}_k : \qquad S_k = \frac{4\pi}{g^2} \times (\mu_{k+1} - \mu_k) = \frac{S_I}{N} \qquad , \qquad k = 1, \ldots, N$$



Small-2d BPST instanton in CP(2)



Large-2d BPST instanton in CP(2) fractionates into 3-types of kink-instantons. (In thermal case, this does not occur at high-T, see Affleck(80s).)

Gauge theory counter-part on $R_3 \times S_1$: Monopole-instantons (caloron constituents) **: van Baal, Kraan, (97/98), Lee-Yi (97)**

Application: N=(2,2) CP(N-1)

In theories with fermions, each kink-event carries fermionic zero mode as per an index theorem, applying Nye-Singer (2000), Poppitz-MÜ(2009).

In supersymmetric theory, each kink-instanton carries two-zero modes, and there are N-types of elementary kink events. Recall that BPST instanton has 2N zero modes, and this fits nicely with the idea of fractionalization.

The kink-amplitudes generate a superpotential. This superpotential, obtained in the compactified theory, by working out the duality in simple quantum mechanics, is identical to the result by Hori and Vafa (2000), obtained by using mirror symmetry on R2.

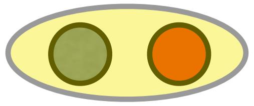
Here, I will address the dynamics of most general CP(N-1), not restrict myself to supersymmetry.

Topological molecules: 2-defects

2-defects are universal, dictated by Cartan matrix of Lie algebra: Charged and neutral bions

Charged bions: For each negative entry of the extended Cartan matrix $\widehat{A}_{ij} < 0$, there exists a bion $\mathcal{B}_{ij} = [\mathcal{K}_i \overline{\mathcal{K}}_j]$, associated with the correlated tunneling-anti-tunneling event

$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_j \qquad \alpha_i \in \Gamma_r^\vee$$



Neutral bions: For each positive entry of the extended Cartan matrix $\widehat{A}_{ii} > 0$, there exists a neutral bion $\mathcal{B}_{ii} = [\mathcal{K}_i \overline{\mathcal{K}}_i]$, associated with the correlated tunneling-anti-tunneling event

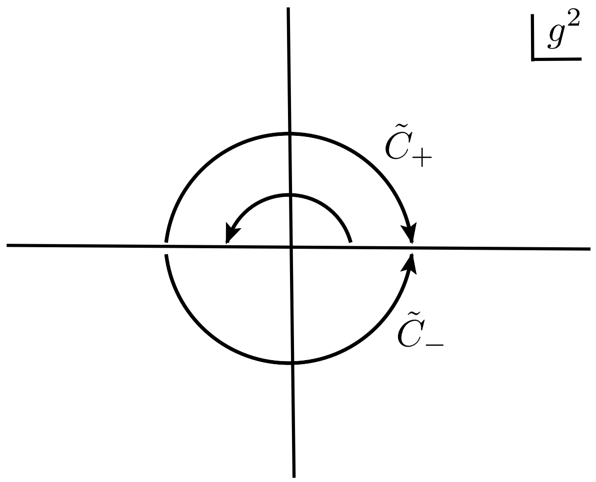
$$\widetilde{n} \longrightarrow \widetilde{n} + \alpha_i - \alpha_i \qquad \alpha_i \in \Gamma_r^\vee$$

Charged bion: Counter-part of magnetic bion in gauge theory on R3 x S1 (generates mass gap for gauge fluctuations), MÜ 2007

Neutral bion is the counter-part of neutral bion in gauge theory on R3 x S1 (generates a center-stabilizing potential), Poppitz-MÜ 2011, Poppitz-Schäfer-MÜ, Argyres-MÜ 2012

Neutral bion and non-perturbative ambiguity in semi-classical expansion

Naive calculation of neutral bion amplitude, as you may guess as per QM example, meaningless at $g^2>0$. The quasi-zero mode integral is dominated at small-separations where a molecular event is meaningless. Continue to $g^2<0$, evaluate the integral there, and continue back to $g^2>0$. Result is two fold-ambiguous!



$$\begin{aligned}[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} &= \mathrm{Re}\,[\mathcal{K}_i\overline{\mathcal{K}}_i] + i\,\mathrm{Im}\,[\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm} \\ &= \left(\log\left(\frac{\lambda}{8\pi}\right) - \gamma\right)\frac{16}{\lambda}e^{-2S_0} \pm i\frac{16\pi}{\lambda}e^{-\frac{8\pi}{\lambda}}\end{aligned}$$

As it stands, this is a **disaster!** Semi-classical expansion at second order is void of meaning? This is a general statement valid for many QFTs admitting semi-classical approximation. e.g. the Polyakov model....

In QFT literature, people rarely discussed second or higher order effects in semi-classics, most likely, they thought no new phenomena would occur, and they would only calculate exponentially small subleading effects. The truth is far more subtler!

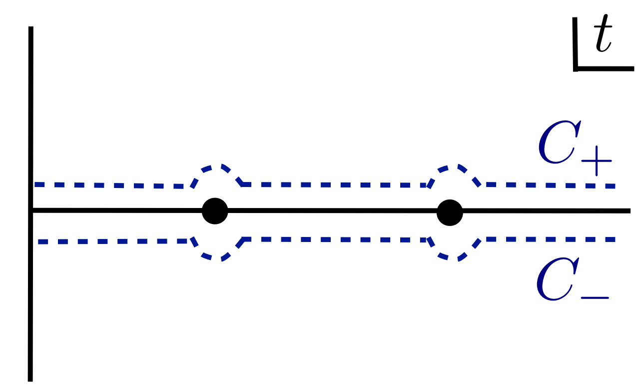
Disaster or blessing in disguise?

Go back to pert. theory, for the compactified center-symmetric CP(N-1) theory. We reduce the long-distance effective theory to simple QM with periodic potentials. Thankfully, the large-order behavior of pert. theory in such QM problems is studied by M. Stone and J. Reeve (78), by using the classic Bender-Wu analysis (69-73).

$$\mathcal{E}(g^2) \equiv E_0 \xi^{-1} = \sum_{q=0}^{\infty} a_q (g^2)^q, \qquad a_q \sim -\frac{2}{\pi}\left(\frac{1}{4\xi}\right)^q q! \left(1 - \frac{5}{2q} + O(q^{-2})\right)$$

Divergent non-alternating series, non-Borel summable, but right and left Borel resummable, with a result:

$$\mathcal{S}_{0\pm}\mathcal{E}(g^2) = \frac{1}{g^2}\int_{C_\pm} dt\, B\mathcal{E}(t)\, e^{-t/g^2} = \mathrm{Re}\mathcal{S}\mathcal{E}(g^2) \mp i\frac{8\xi}{g^2} e^{-\frac{4\xi}{g^2}}$$
$$= \mathrm{Re}\mathbb{B}_0 \mp i\frac{16\pi}{g^2 N} e^{-\frac{8\pi}{g^2 N}}$$



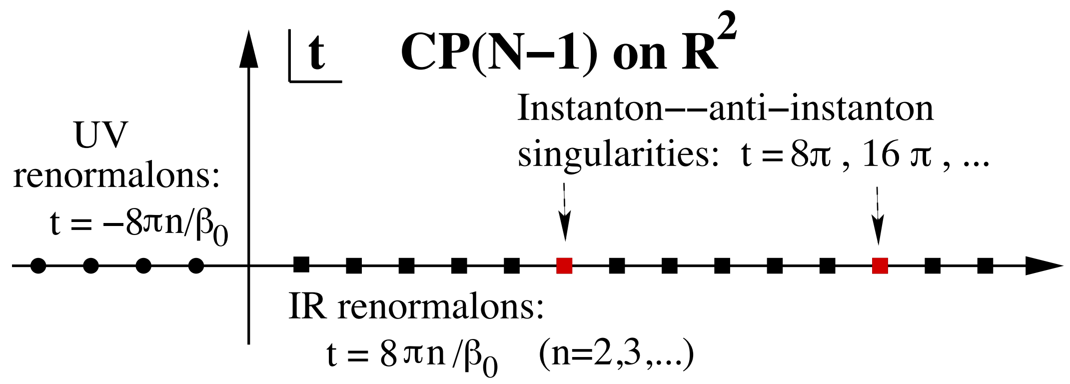
Remarkably,

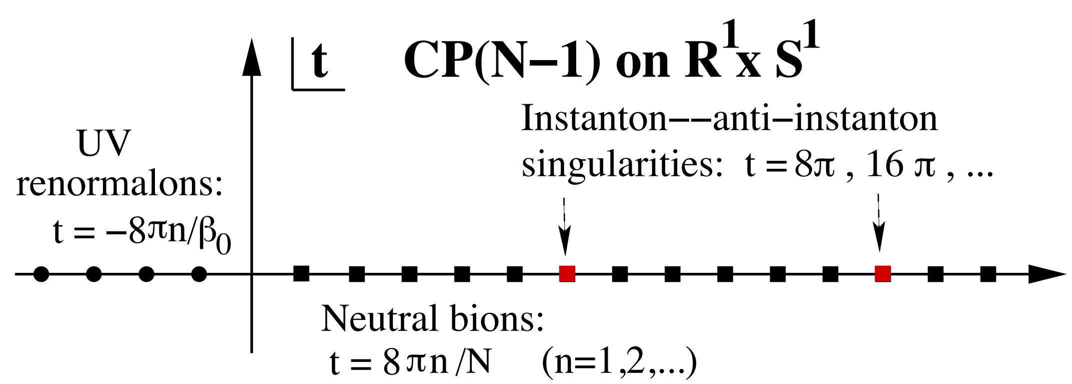
$$\mathrm{Im}\Big[\mathcal{S}_\pm\mathcal{E}(g^2) + [\mathcal{K}_i\overline{\mathcal{K}}_i]_{\theta=0^\pm}\Big] = 0 \qquad \text{up to } e^{-4S_0} = e^{-4S_I/\beta_0}$$

The ambiguities at order exp[-2S$_I$/N] cancel and
QFT is well-defined up to the ambiguities of order exp[-4S$_I$/N] ! Ambiguities exactly in the IR-renormalon territory as per ‘t Hooft, David.

Semi-classical renormalons as neutral bions

Claim (with Argyres in 4d) and (with Dunne in 2d): Neutral bions and neutral topological molecules are semi-classical realization of 't Hooft's elusive renormalons, and it is possible to make sense out of combined perturbative semi-classical expansion. We showed this only at leading (but most important) order. Subleading orders underway.





More than three decades ago, 't Hooft gave a famous set of (brilliant) lectures(79): *Can we make sense out of QCD?* He was thinking a non-perturbative continuum formulation. It seem plausible to me that in fact, we can, at least, in the semi-classical regime of QFT. (and perhaps even more.)

Why is this happening? Stokes, Stokes and more Stokes.....

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments. By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]

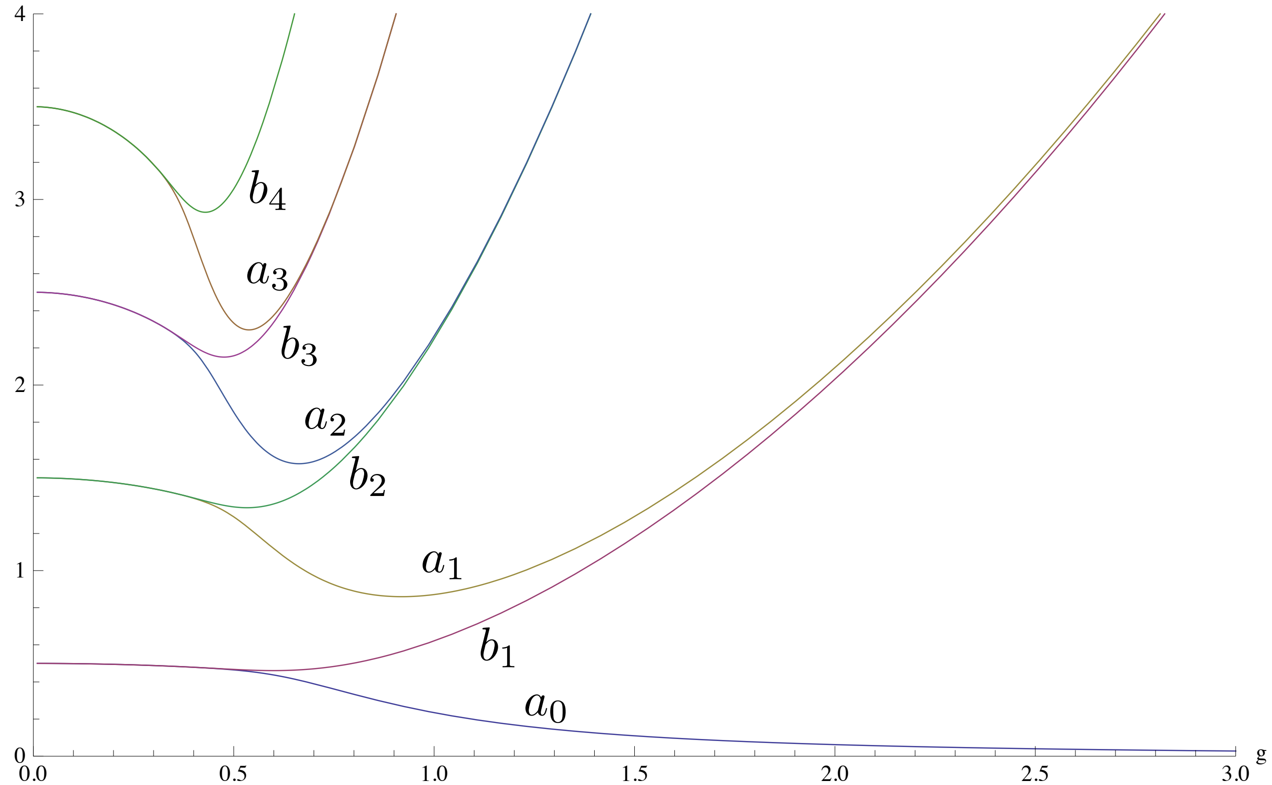
IN a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large.

Perturbation theory: non-alternating divergent asymptotic expansion, and ambiguity: Because we are doing an expansion on a **Stokes line/wall.** **This should have been a standard QFT/QM textbook material since many decades. It is sad that it is not. Not many ask questions about the meaning of perturbation theory.**

Stokes phemomenon is a "super-subtlety", not much understood by his contemporaries, even today, not sufficiently appreciated.

Ambiguity in neutral bion: Mirroring the Stokes jump as one moves from one Stokes chamber to another (a version of wall-crossing).

Mass gap in the small-S1 regime



Eigen-energies for CP(1) in reduced QM. In Born-Oppenheimer approx., the zero mode Hamiltonian at small g reduce to Mathieu ODE.

$$H^{\text{zero}}_{\alpha_k} = -\frac{1}{2}\frac{d^2}{d\theta^2} + \frac{\xi^2}{4g^2}[1-\cos(2g\theta)]$$

$$m_g = \frac{C}{\sqrt{\lambda}}\left(1 - \frac{7\lambda}{32\pi} + O(\lambda^2)\right) e^{-\frac{4\pi}{\lambda}} \sim e^{-S_I/N} \qquad \text{for } \mathbb{CP}^{\text{N}-1}$$

The functional form of the small-S1 result for CP(N-1) is same as large-N result on R2. This is the first microscopic derivation of the factor exp[-S_I/N] from microscopic considerations. This effect, at least in the small-S1 regime, solves the large-N vs. instanton puzzle. Clearly, this is an effect which survives large-N limit.

Resurgence Theory and Transseries

Ecalle (1980s) formalized asymptotic expansion with exponentially small terms (called trans-series) & generalized Borel resummation for them by incorporating the Stokes phenomenon. Grand generalization of Borel-summability, a way to deal with non-Borel summable series...

Basic idea: Start with a formal power series, e.g. an asymptotic (divergent) expansion of Gevrey-1 type: $\Sigma\, a_n g^{2n}$ where $a_n \leq A\, n!\, c^n$ (generic in QFT).

Borel transformation maps this formal series to a convolutive subalgebra of germs (geometric series with a finite radius of divergence) at the origin in Borel plane.

Analytic continuation of the germ to a holomorphic function except a set of singularities (pole or branch points) in the complex Borel-plane.

Directional Laplace transforms to find sectorial sums by invoking Stokes phenomenon.

Resurgence theory of Ecalle

Main result: Borel-Ecalle resummation of a transseries exists and is unique, if the Borel transforms of all perturbative series are all "endlessly continuable" =Set of all singularities on all Riemann sheets on Borel plane do not form any natural boundaries.

Such transseries are called "resurgent functions": Example of transseries:

$$f(\lambda\hbar) \sim \sum_{k=0}^{\infty} c_{(0,k)} \, (\lambda\hbar)^k + \sum_{n=1}^{\infty} (\lambda\hbar)^{-\beta_n} \, e^{-n\, A/(\lambda\hbar)} \sum_{k=0}^{\infty} c_{(n,k)} \, (\lambda\hbar)^k$$

Formal: perturbative + (non-perturbative) x (perturbative)

Resurgence theory of Ecalle in QFTs

Pham, Delabaere,....(1990s): Using the theory of resurgent functions, they proved that the semi-classical (perturbative+ non-perturbative) transseries expansion in Quantum mechanics with double-well and periodic potentials are summable to finite, exact results.

In CP(N-1), by invoking "continuity", we can reduce QFTs in long distance limit to the quantum mechanical systems studied by Pham et.al. In particular, Pham et.al. result implies that in the small S1 regime, spectrum, mass gap etc. of the theory are resurgent functions. What I showed you was one of the first step of cancellations inherent to resurgence theory applied to QFT.

Resurgence theory in path integrals

Pham et. al. results are in Hamiltonian formalism. We wonder whether we can generalize this to path integrals of QM, because, path integral formulation generalize more easily to QFT.

Key step is in the analytic continuation of paths in field space (cf. Pham, and recent papers by Witten), to make sense of steepest descent and Stokes phenomenon in path integrals. (We actually use this implicitly, but need to make it more systematic.)
cf. a recent talk by Kontsevich "Resurgence from the path integral perspective", Perimeter Institute, August, 2012.

Recent work: Basar, Dunne, MU d=0 and d=1 systems, not in this talk
Argyres, MU: QM in path integrals, similar to Kontsevich's ideas.

Graded Resurgence triangle

The structure of CP(N-1) and many QFTs is encoded into the following construct:

$f_{(0,0)}$ → pert. theory around pert. vacuum

kink x (pert. fluctuations) ← $e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)}$ $\quad e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)}$

$e^{-\frac{2A}{\lambda}} f_{(2,0)}$ → bions x (pert. fluctuations)

$$
\begin{array}{ccccccccc}
 & & & & f_{(0,0)} & & & & \\
 & & & e^{-\frac{A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(1,1)} & & e^{-\frac{A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(1,-1)} & & & \\
 & & e^{-\frac{2A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,2)} & & e^{-\frac{2A}{\lambda}} f_{(2,0)} & & e^{-\frac{2A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(2,-2)} & & \\
 & e^{-\frac{3A}{\lambda}+3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,3)} & & e^{-\frac{3A}{\lambda}+i\frac{\widetilde{\Theta}_k}{N}} f_{(3,1)} & & e^{-\frac{3A}{\lambda}-i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-1)} & & e^{-\frac{3A}{\lambda}-3i\frac{\widetilde{\Theta}_k}{N}} f_{(3,-3)} & \\
e^{-\frac{4A}{\lambda}+4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,4)} & & e^{-\frac{4A}{\lambda}+2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,2)} & & e^{-\frac{4A}{\lambda}} f_{(4,0)} & & e^{-\frac{4A}{\lambda}-2i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-2)} & & e^{-\frac{4A}{\lambda}-4i\frac{\widetilde{\Theta}_k}{N}} f_{(4,-4)} \\
 & & \begin{array}{c}\cdot\,^{\cdot}\,^{\cdot}\end{array} & & \vdots & & \ddots & &
\end{array}
$$

No two column can mix with each other in the sense of cancellation of ambiguities.

Non-perturbative saddles are not classified by homotopy. Any two cell in a given column are homotopically identical.

Resurgent (much more refined) vs. topological classification.

N.P. confluence equations

In order QFT to have a meaningful semi-classical continuum definition, a set of perturbative--non-perturbative confluence equations must hold. Examples are

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm} + \mathbb{B}_{[2,0],\theta=0^\pm}[\mathcal{B}_{ii}]_{\theta=0^\pm} + \mathbb{B}_{[4,0],\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_{\theta=0^\pm} + \mathbb{B}_{[6,0]\theta=0^\pm}[\mathcal{B}_{ij}\mathcal{B}_{jk}\mathcal{B}_{ki}]_{\theta=0^\pm} + \dots\Big)$$

Meaning, order by order hierarchical confluence equations:

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm\ , \qquad (\text{up to } e^{-4S_0})$$

$$0 = \mathrm{Im}\mathbb{B}_{[0,0]\pm} + \mathrm{Re}\mathbb{B}_{[2,0]}\mathrm{Im}[\mathcal{B}_{ii}]_\pm + \mathrm{Im}\mathbb{B}_{[2,0]\pm}\mathrm{Re}[\mathcal{B}_{ii}] + \mathrm{Re}\mathbb{B}_{[4,0]}\mathrm{Im}[\mathcal{B}_{ij}\mathcal{B}_{ji}]_\pm \qquad (\text{up to } e^{-6S_0})$$

$$0 = \dots$$

This has a very deep implication in QFT:

Decoding late terms in pert. theory.

$$\operatorname{Disc} \mathbb{B}_{[0,0]} = -2\pi i \lambda^{-r_2} P_{[2,0]} e^{-2A/\lambda} + \mathcal{O}(e^{-4A/\lambda}), \tag{1}$$

Using dispersion relation, we obtain

$$\begin{aligned} a_{[0,0],q} &= \sum_{q'=0}^{\infty} a_{[2,0],q'} \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2-q'}} + O\left(\left(\frac{1}{4A}\right)^q\right) \\ &= \frac{\Gamma(q+r_2-q')}{(2A)^{q+r_2}} \left[a_{[2,0],0} + \frac{2A}{(q+r_2-1)} a_{[2,0],1} + \frac{(2A)^2}{(q+r_2-1)(q+r_2-2)} a_{[2,0],2} + \ldots \right] \\ &+ O\left(\left(\frac{1}{4A}\right)^q\right) \end{aligned} \tag{2}$$

Late terms in pert.exp. around the pert. vac.

Early terms in pert.exp. around neutral bion= 1/q corrections:

Neutral bion action

Exponentially suppressed corrections: Bion-bion etc. terms.

Why extended supersymmetric theories are simpler?

$$
\begin{array}{ccccccccc}
 & & & & [0,0] & & & & \\
 & & & [1,1] & & [1,-1] & & & \\
 & & [2,2] & & \varnothing & & [2,-2] & & \\
 & [3,3] & & \varnothing & & \varnothing & & [3,-3] & \\
[4,4] & & \varnothing & & \varnothing & & \varnothing & & [4,-4] \\
 & & & & \vdots & & & & \ddots
\end{array}
$$

No neutral bion configurations, the confluence equation simplify into

$$0 = \mathrm{Im}\Big(\mathbb{B}_{[0,0],\theta=0^\pm}\Big), \qquad 0 = \mathrm{Im}\Big(\mathbb{B}_{[1,1],\theta=0^\pm}\Big)$$

Extended supersymmetric theories must be Borel summable!

Argyres, MÜ,
Dunne, MÜ, 2012

Same conclusion, Russo (2012) by explicit computation using localization.

Conclusions

Continuity and resurgence theory can be used in combination to provide a non-perturbative continuum definition of asymptotically free theories, and more general QFTs.

Resurgence provides a more refined classification of non-perturbative saddles with respect to topological classification. In particular, it can distinguish NP-contributions even when they are in the same homotopy class.

The construction may have practical utility and region of overlap with lattice field theory. One can check predictions of the formalism numerically.