Progress in calculating multiparticle amplitudes from lattice QCD

Steve Sharpe
University of Washington

Dreams of a lattice theorist

Testing the SM with D decays

Observation of CP violation in charm decays

CERN-EP-2019-042
13 March 2019

LHCb collaboration†

Abstract

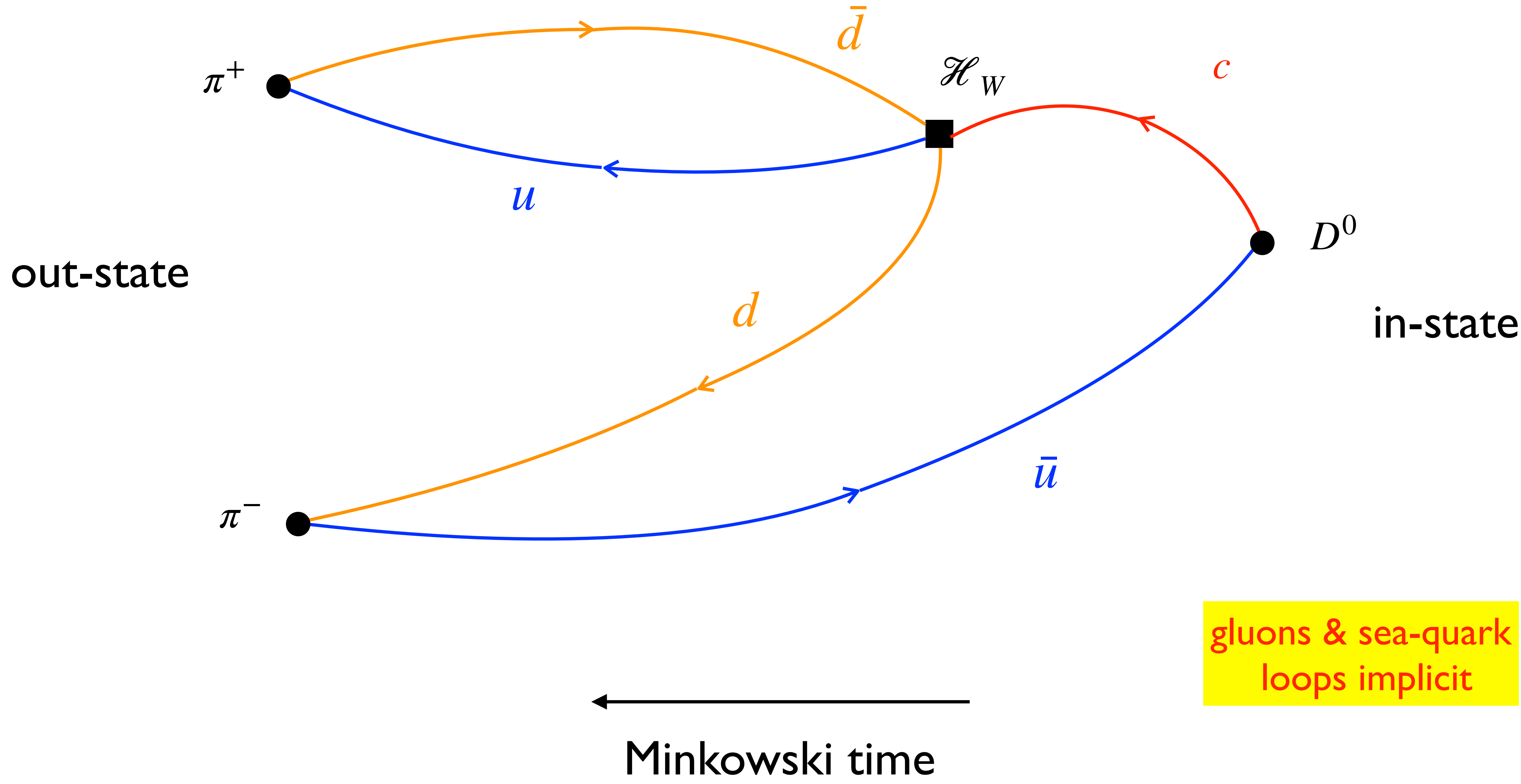
A search for charge-parity (CP) violation in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is reported, using pp collision data corresponding to an integrated luminosity of $6\,\text{fb}^{-1}$ collected at a center-of-mass energy of $13\,\text{TeV}$ with the LHCb detector. The flavor of the charm meson is inferred from the charge of the pion in $D^*(2010)^+ \to D^0 \pi^+$ decays or from the charge of the muon in $\overline{B} \to D^0 \mu^- \bar{\nu}_\mu X$ decays. The difference between the CP asymmetries in $D^0 \to K^- K^+$ and $D^0 \to \pi^- \pi^+$ decays is measured to be $\Delta A_{CP} = [-18.2 \pm 3.2\,(\text{stat.}) \pm 0.9\,(\text{syst.})] \times 10^{-4}$ for π-tagged and $\Delta A_{CP} = [-9 \pm 8\,(\text{stat.}) \pm 5\,(\text{syst.})] \times 10^{-4}$ for μ-tagged D^0 mesons. Combining these with previous LHCb results leads to

$$\Delta A_{CP} = (-15.4 \pm 2.9) \times 10^{-4},$$

where the uncertainty includes both statistical and systematic contributions. The measured value differs from zero by more than five standard deviations. This is the first observation of CP violation in the decay of charm hadrons.

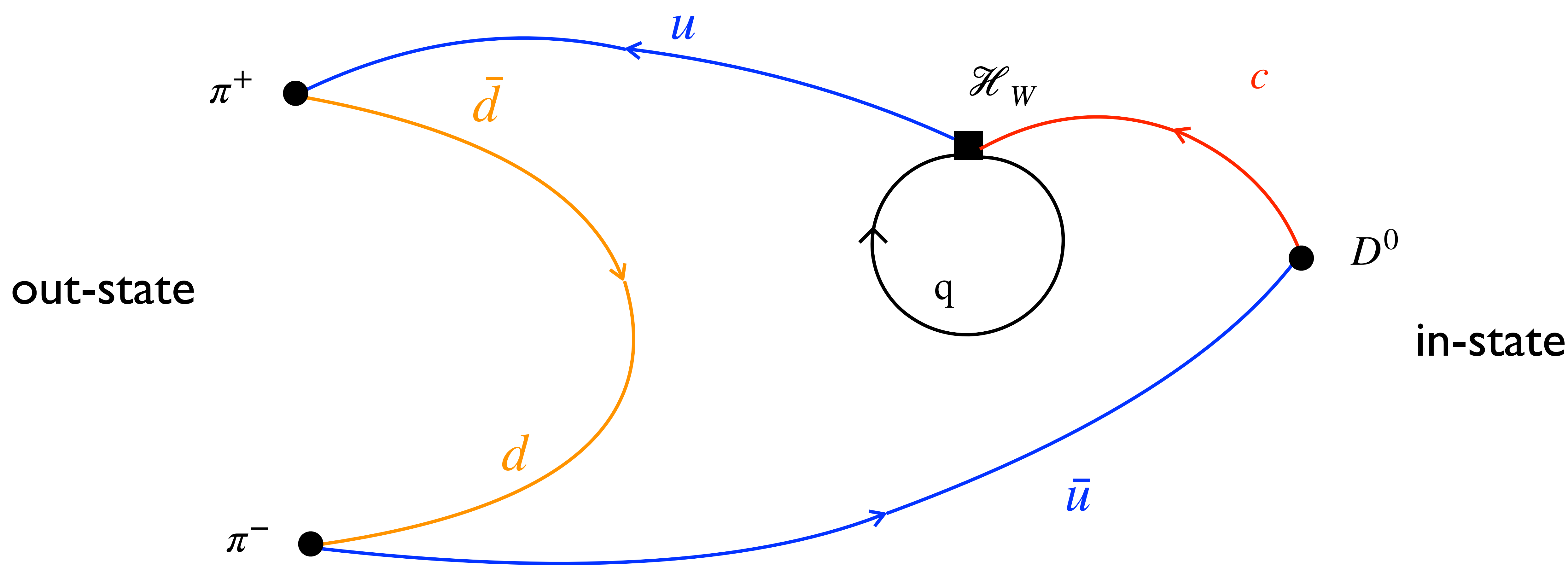
5.3σ effect

What we want…



gluons & sea-quark loops implicit

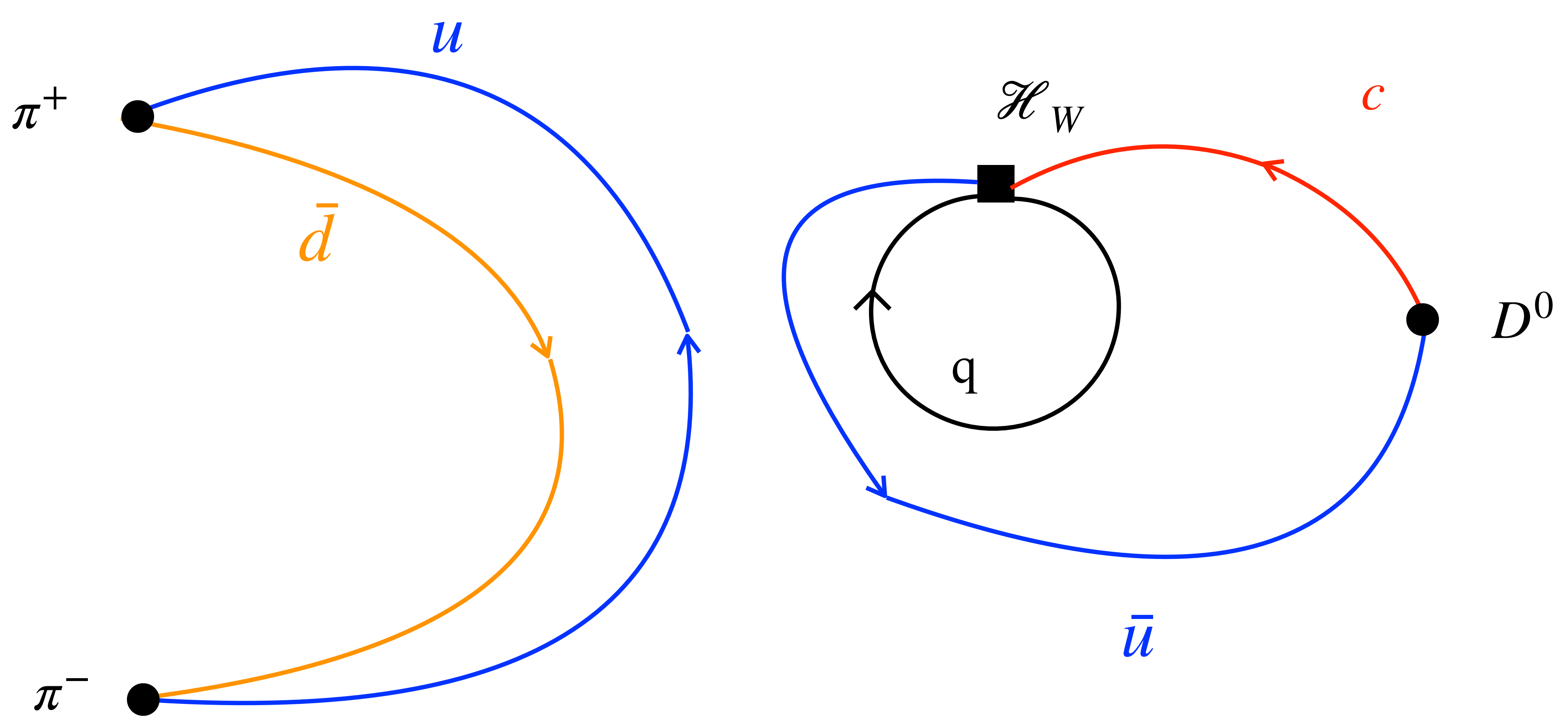
What we want…



gluons & sea-quark loops implicit

Minkowski time

What we want…



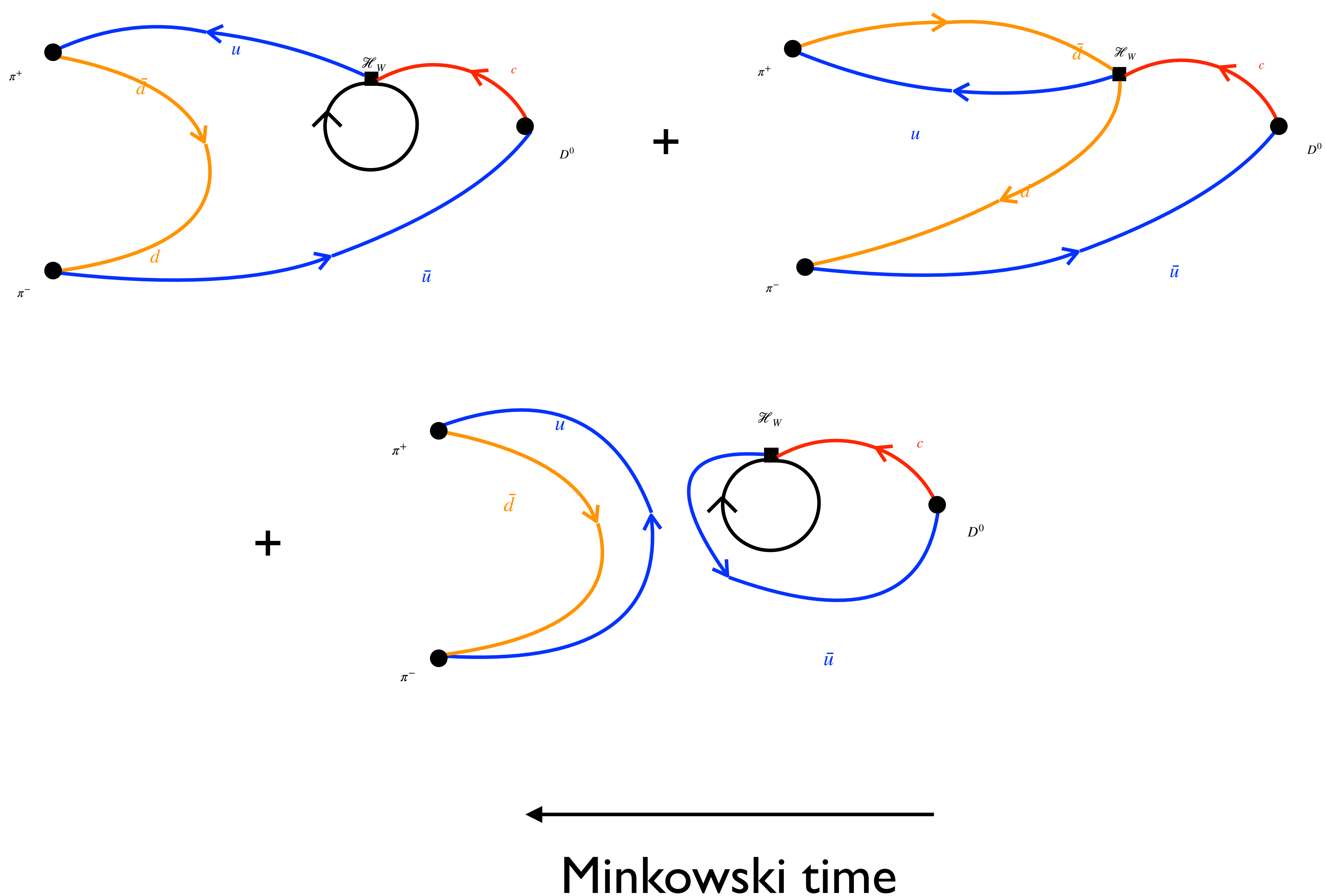
out-state

in-state

gluons & sea-quark loops implicit

Minkowski time

What we want…


u
$\bar{d}$
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π^-
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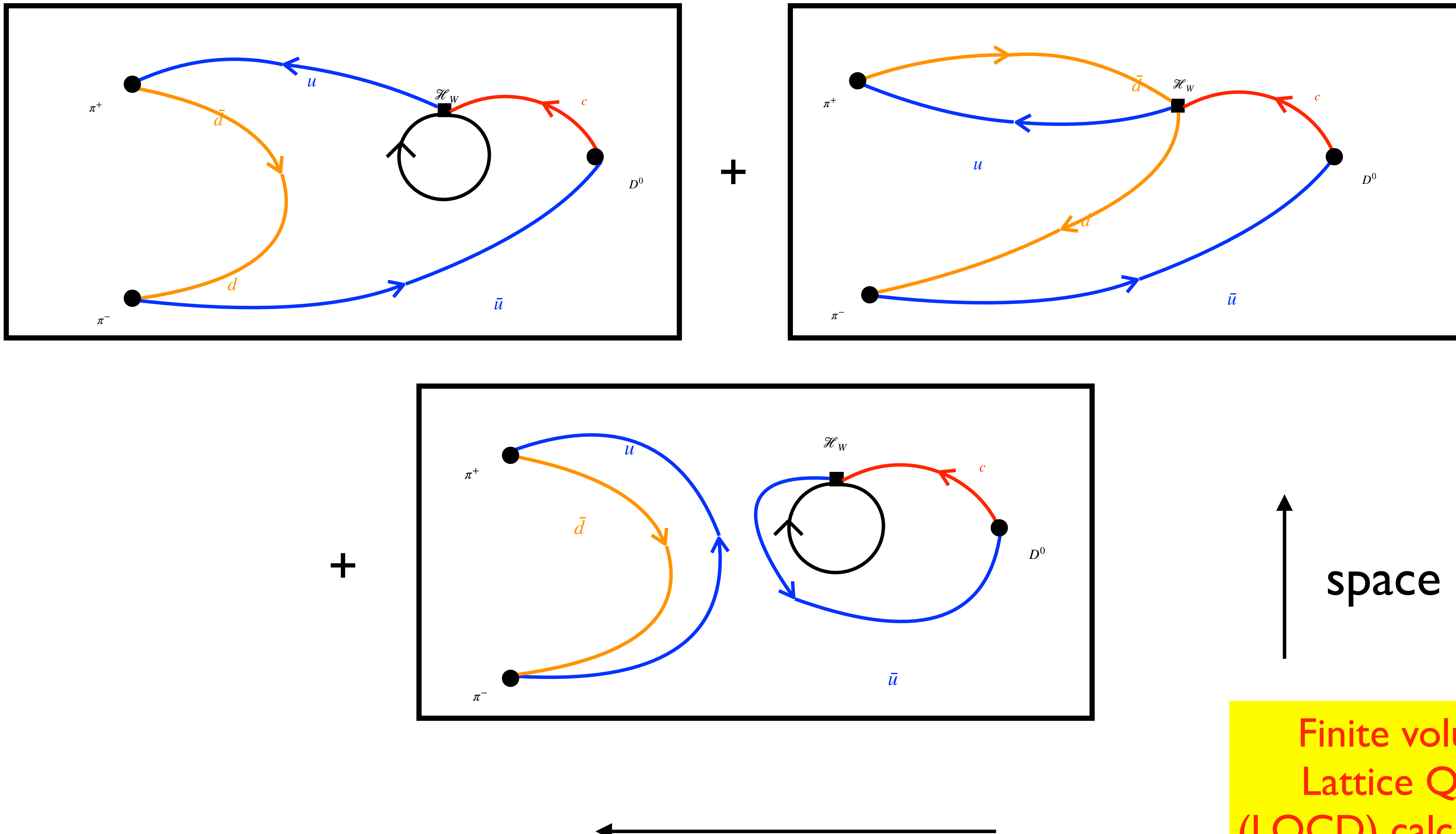
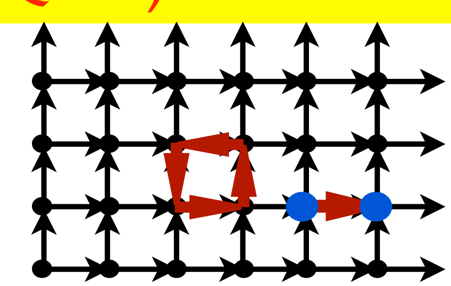
Minkowski time

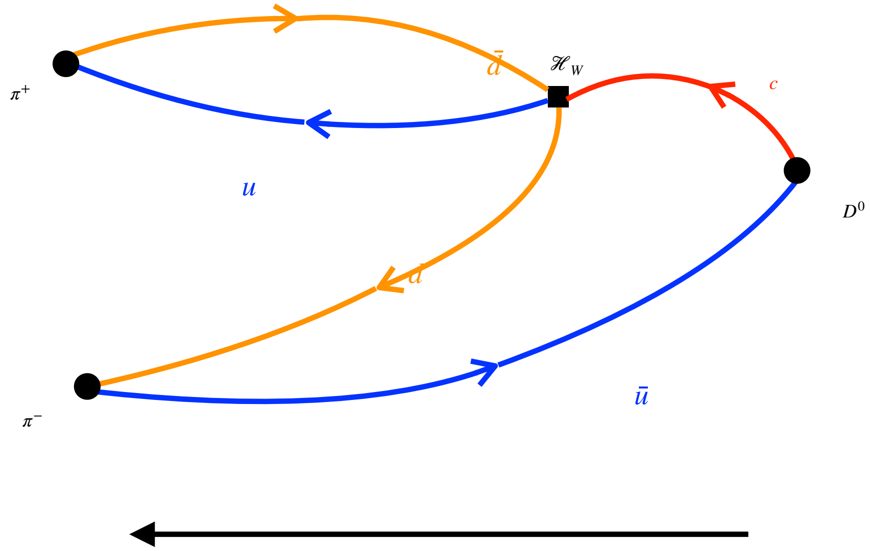
...what we might achieve



Finite volume
Lattice QCD
(LQCD) calculations

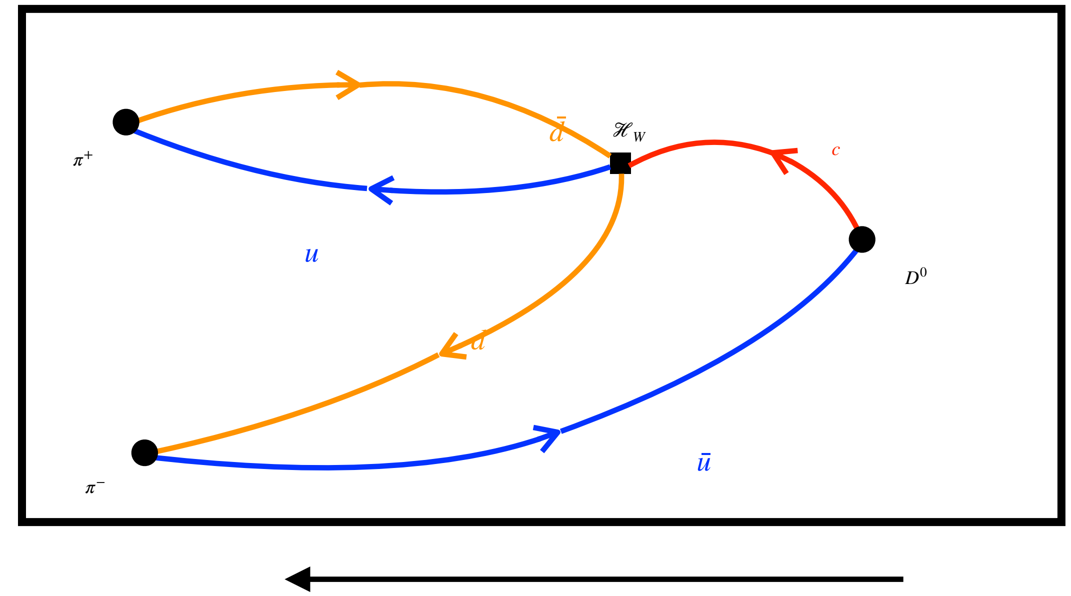


Problems



Minkowski time

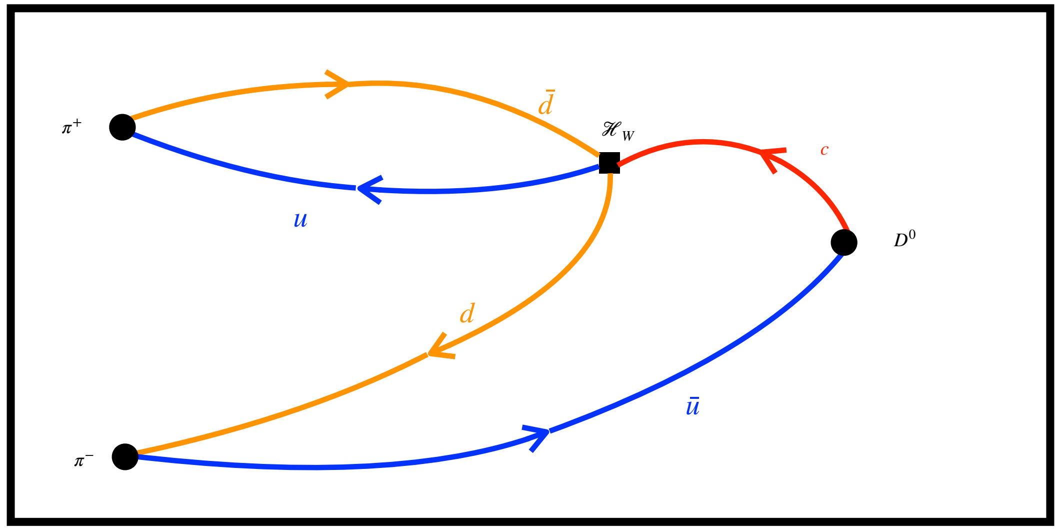
vs.



Euclidean time

- No in- and out-states in finite volume
  - Cannot separate final-state particles
- Need to analytically continue from Euclidian to Minkowski momenta
  - Ill-posed problem given discrete momenta in finite volume

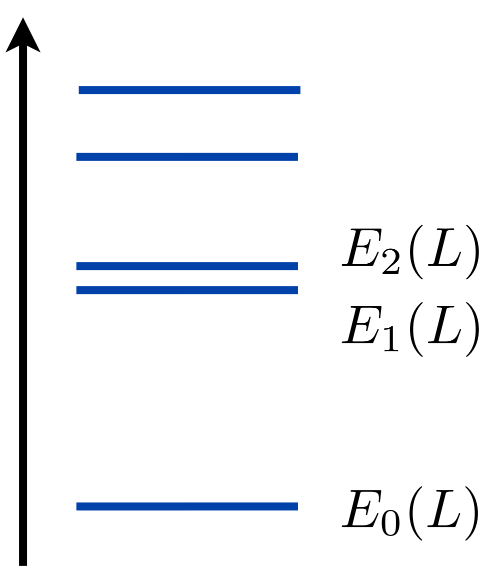
Rephrasing



space

Euclidean time

What LQCD can determine are finite-volume matrix elements:

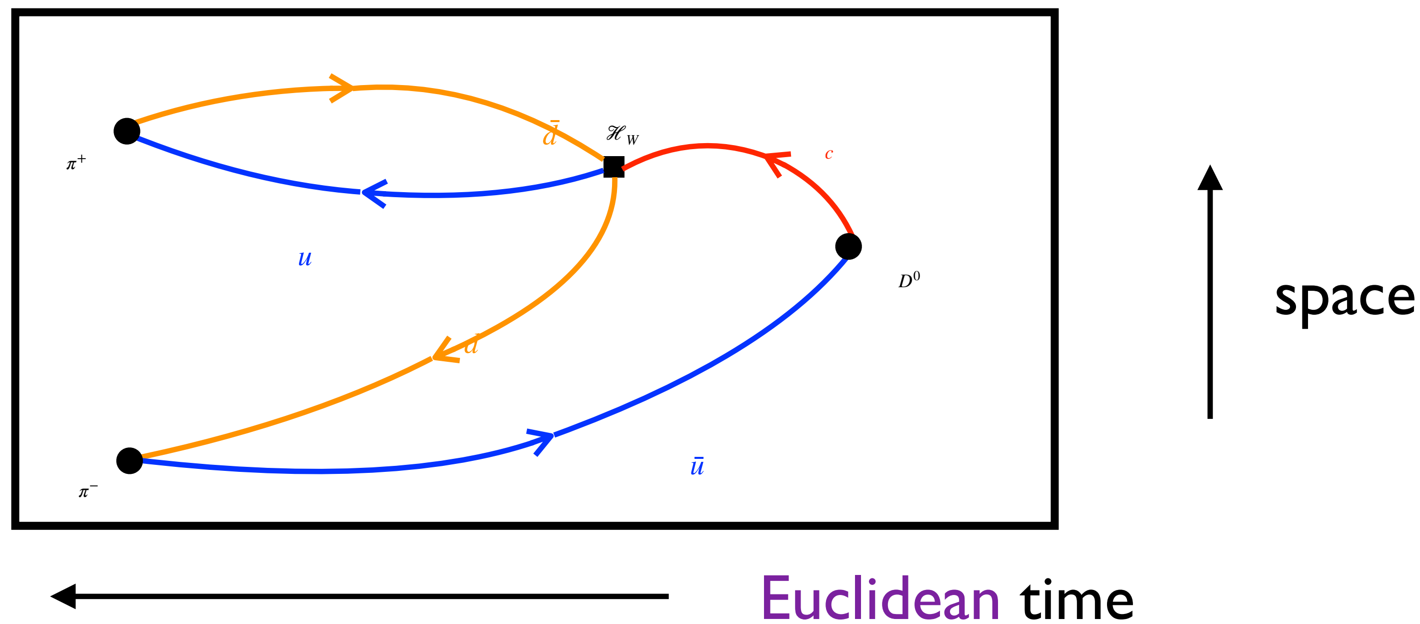


$$_L\langle E_n | \mathscr{H}_W | D^0 \rangle_L$$

Discrete spectrum of states with quantum numbers of $\pi^+\pi^-$

Physical quantities if choose $E_n(L) = E_D$

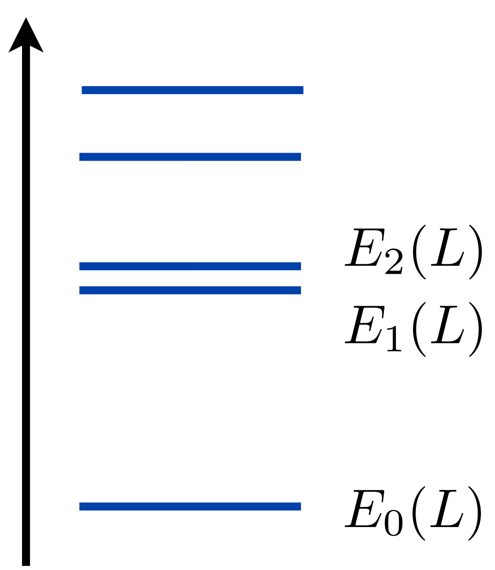
Rephrasing



What LQCD can determine are finite-volume matrix elements:

$${}_L\langle E_n | \mathscr{H}_W | D^0 \rangle_L$$

Physical quantities if choose $E_n(L) = E_D$



Discrete spectrum of states with quantum numbers of $\pi^+\pi^-$

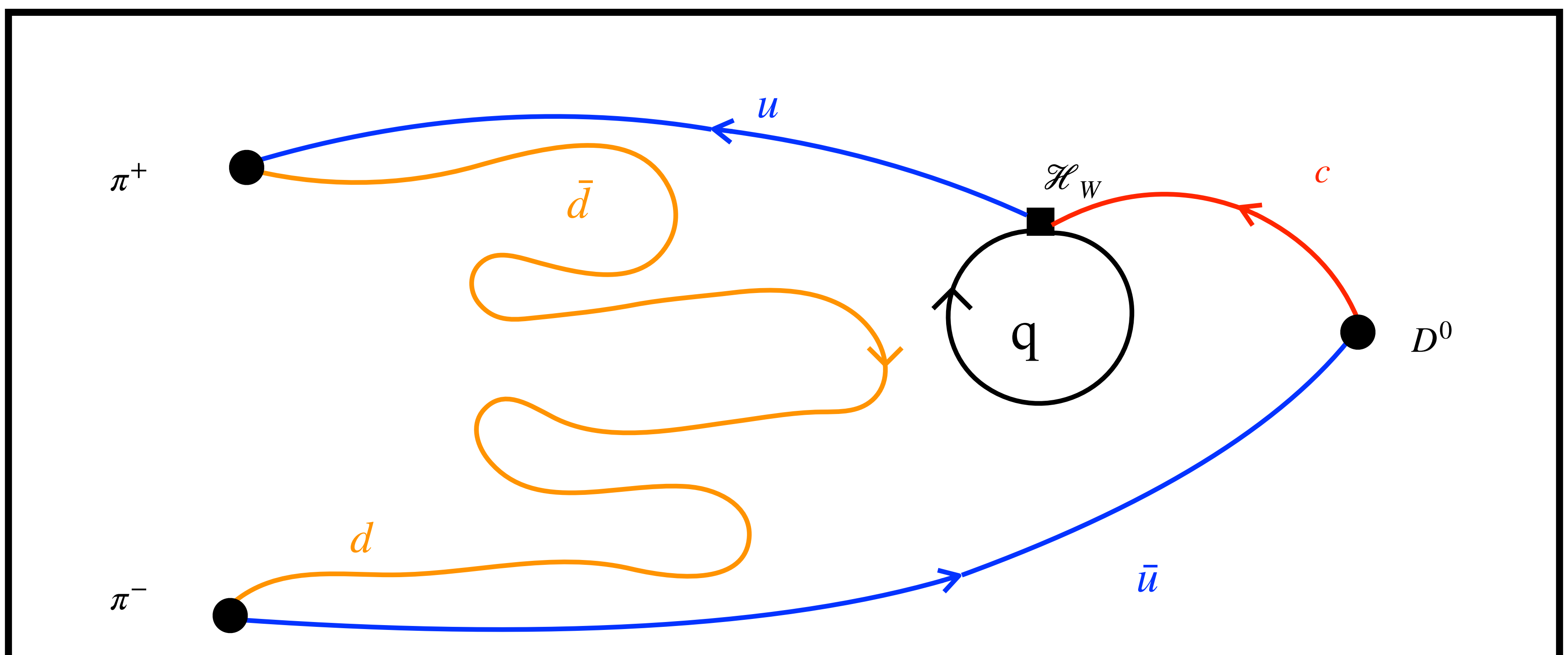
LQCD methods could, in the near future, allow the calculation of these quantities

How can they be related to the physical decay amplitudes?

The fundamental issue

$$_L\langle E_n | \mathscr{H}_W | D^0 \rangle_L \longrightarrow {}_{\rm out}\langle \pi^+\pi^- | \mathscr{H}_W | D^0 \rangle_{\rm in}$$

- This is a nontrivial (and so-far unsolved) QFT problem because $|E_n\rangle_L$ are composed of contributions from $\pi\pi$, 4π, $K\bar{K}$, 6π, … with $j = 0,2,\ldots$
  - Even if you use a two-pion operator, the strong interactions unavoidably lead to mixing with other states
  - Solution will require amplitudes for $\pi\pi \to \pi\pi$, $3\pi \to 3\pi$, $\pi\pi \to 4\pi$, …, which will need to be determined from the energies $E_n(L)$



The fundamental issue

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- This is a nontrivial (and so-far unsolved) QFT problem because $|E_n\rangle_L$ is composed of contributions from $\pi\pi$, 4π, $K\bar{K}$, 6π, $\ldots$ with $j = 0,2,\ldots$
    - Even if you use a two-pion operator, the strong interactions unavoidably lead to mixing with other states
    - A solution will require amplitudes for $\pi\pi \to \pi\pi$, $3\pi \to 3\pi$, $\pi\pi \to 4\pi$, $\ldots$, which will need to be determined from the energies $E_n(L)$
    - A side benefit of any solution will be the ability to use LQCD results for $E_n(L)$ to study resonances with decays into multiple two-, three- and four-particle channels

Over the last 10 years, the corresponding issues for three particles have been solved, and are beginning to be implemented in LQCD simulations

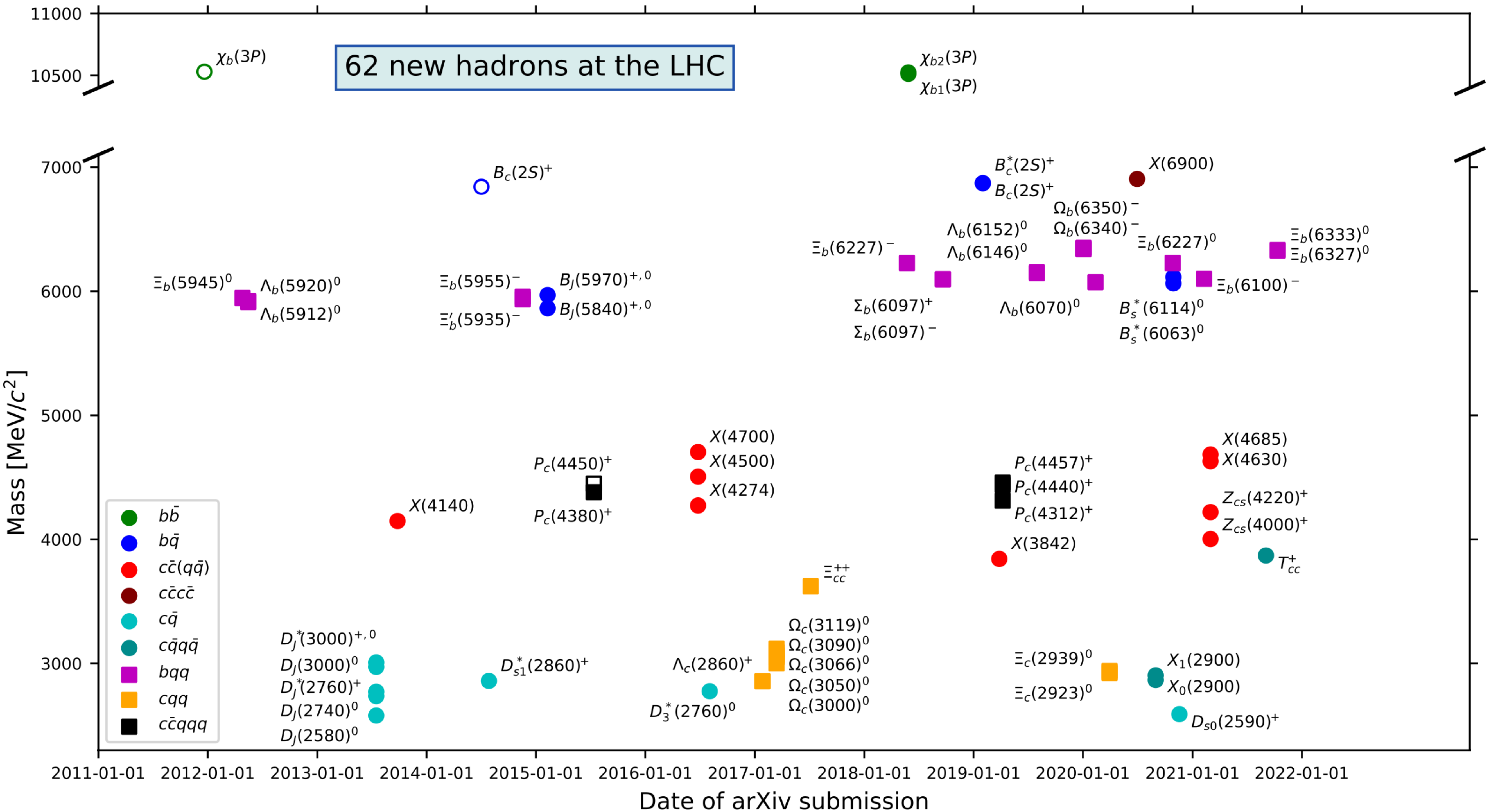
Today I will briefly summarize the status, and describe examples of recent work

Outline

- Further motivation for studying 3-particle resonances & decays
- History and status of finite-volume formalism for 2 & 3 particles
- Examples of recent work
    - Extraction of $\pi^+\pi^+K^+ \to \pi^+\pi^+K^+$ and $K^+K^+\pi^+ \to K^+K^+\pi^+$ K-matrices using LQCD with close to physical quark masses
    - NLO Chiral PT calculation of three-particle K matrix
- Summary & Outlook

Motivations for studying three particles using LQCD

Cornucopia of exotics



[I. Danilkin, talk at INT workshop, March 23]

+ data from Babar, Belle, COMPASS, …

Motivations

- Most resonances have 3 (or more) particle decay channels
  - $\omega(782,\, I^G J^{PC} = 0^- 1^{--}) \to 3\pi$
  - $N(1440,\, J^P = \frac{1}{2}^+) \to N\pi,\, N\pi\pi$
  - $T_{cc}(3875,\, I = 0,\, J^P = 1^+?) \to D^0 D^0 \pi^+$
- Determining 3-body "forces"
  - NNN interactions needed as input for EFT treatments of large nuclei, and for the neutron-star equation of state
  - $\pi\pi\pi, \pi K\bar{K}, \ldots$ interactions needed as input to study pion & kaon condensation

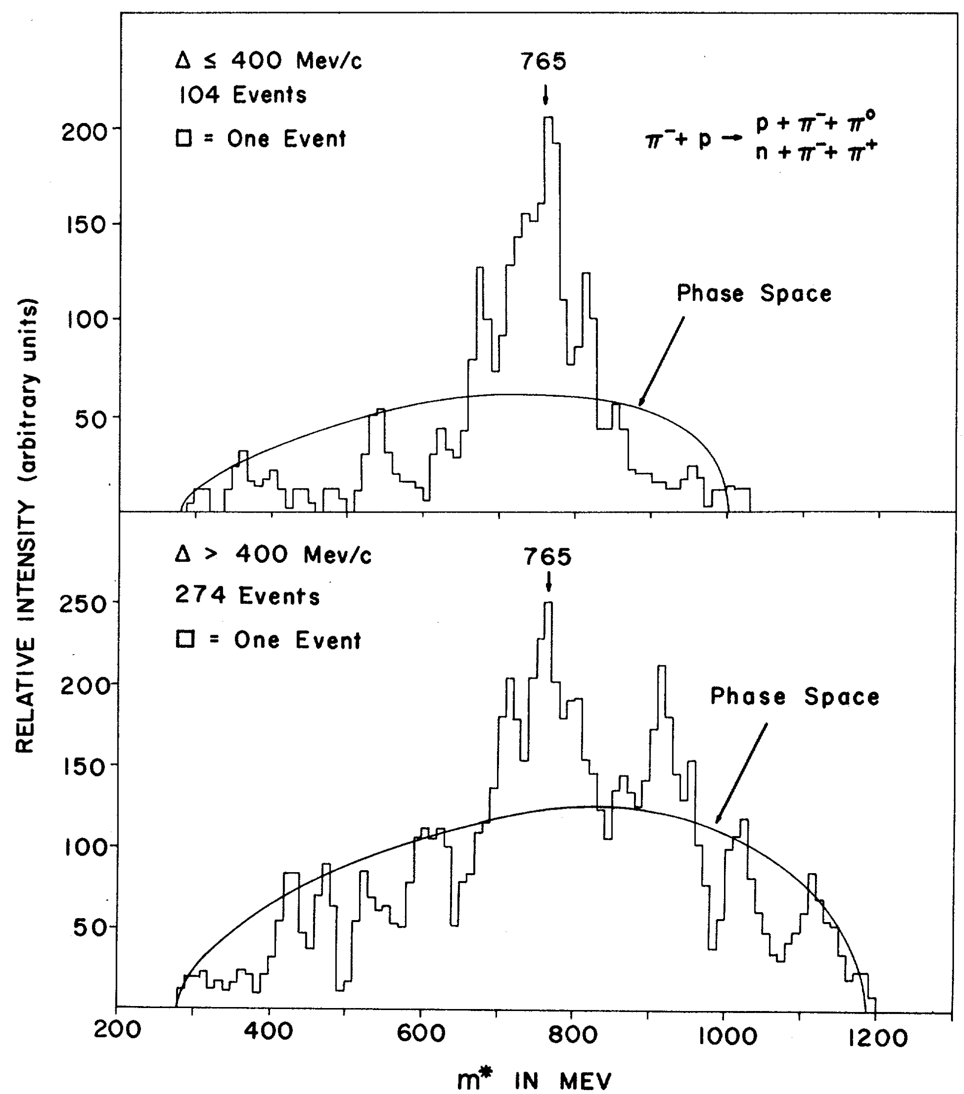
History & status of finite-volume formalism for 2 particles

Sketch of history for two particles

- 1961: Discovery of the ρ meson

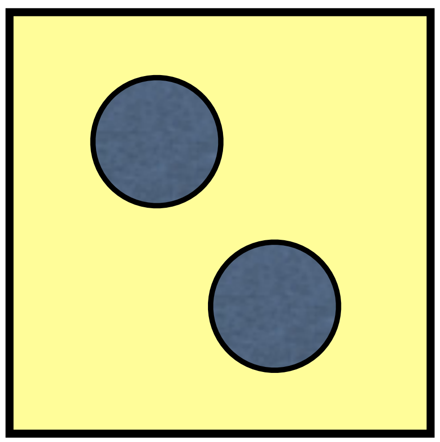
EVIDENCE FOR A π-π RESONANCE IN THE $I=1$, $J=1$ STATE*

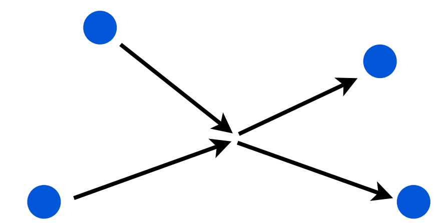
A. R. Erwin, R. March, W. D. Walker, and E. West
Brookhaven National Laboratory, Upton, New York and University of Wisconsin, Madison, Wisconsin
(Received May 11, 1961)



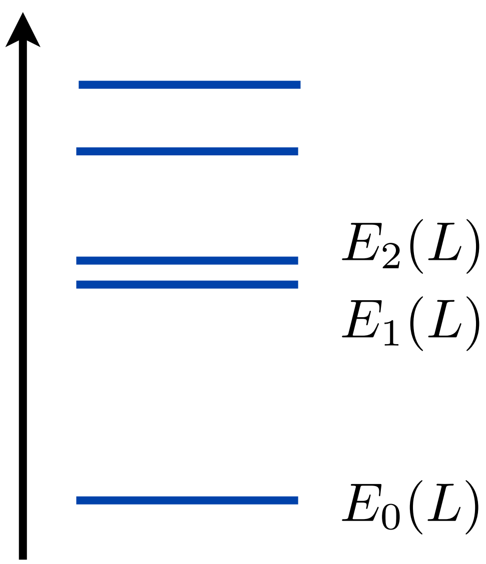
Sketch of history for two particles

- 1961: Discovery of the ρ meson
- 1986/91: Lüscher derived "two-particle quantization condition" (QC2)





Scattering amplitude



Discrete energy spectrum
e.g. $\pi\pi$ with
$E_{\rm CM} < 4M_\pi$

$$\det\left[F(E,\mathbf{P},L)^{-1} + \mathscr{K}_2(E^*)\right] = 0$$

Matrices in ℓ, m

F is a known kinematical "zeta-function", depending on the box shape

$\mathscr{K}_2 \sim \tan\delta/q$

Valid up to e^{-ML} corrections

Sketch of history for two particles

- 1961: Discovery of the ρ meson
- 1986/91: Lüscher derived "two-particle quantization condition" (QC2)
  - 2005: Kim, Sachrajda & SRS — alternate derivation, basis for many subsequent generalizations
- 1999: Measurement of $\varepsilon'/\varepsilon = 16.1(2.3)10^{-4}$ by KTeV/NA48 — direct CPV in $K \to \pi\pi$

NA48 @ CERN: Cern website

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- 2001: Lellouch & Lüscher (LL): relation between ${}_L\langle E_n(\pi\pi) | \mathscr{H}_W | K \rangle_L$ and $\mathscr{A}(K \to \pi\pi)$

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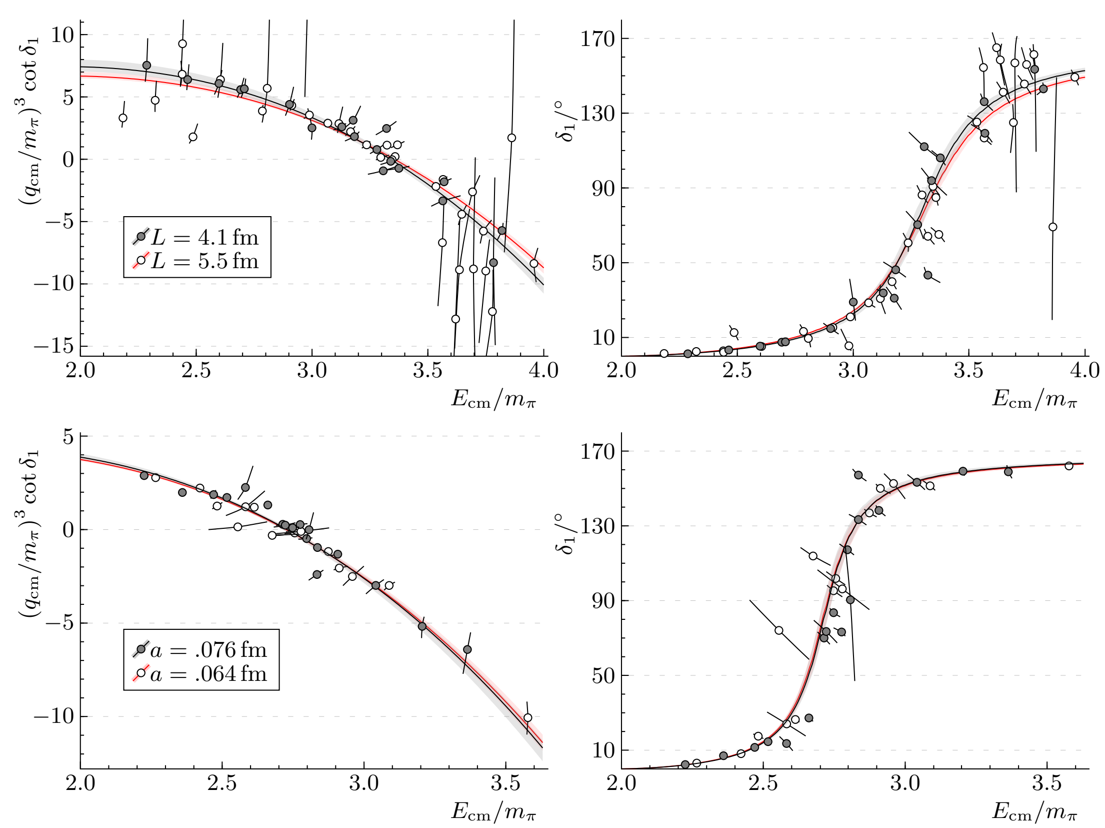
- 2014 - 2019: LQCD implementation of QC2 for the ρ resonance in $\pi\pi$ scattering (and many other resonances subsequently)

Anderson et al., 1808.05007



$M_\pi \approx 200\,\mathrm{MeV}$

$M_\pi \approx 280\,\mathrm{MeV}$

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- 2014 - 2019: LQCD implementation of QC2 for the ρ resonance in $\pi\pi$ scattering
- 2020: LQCD calculation of $\epsilon'/\epsilon = 21.7(8.4)10^{-4}$ in the standard model using LL method with physical quark masses and (almost) all errors controlled

Direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay from the standard model

R. Abbott,[1] T. Blum,[2,3] P. A. Boyle,[4,5] M. Bruno,[6] N. H. Christ,[1] D. Hoying,[3,2] C. Jung,[4] C. Kelly,[4] C. Lehner,[7,4] R. D. Mawhinney,[1] D. J. Murphy,[8] C. T. Sachrajda,[9] A. Soni,[4] M. Tomii,[2] and T. Wang[1]

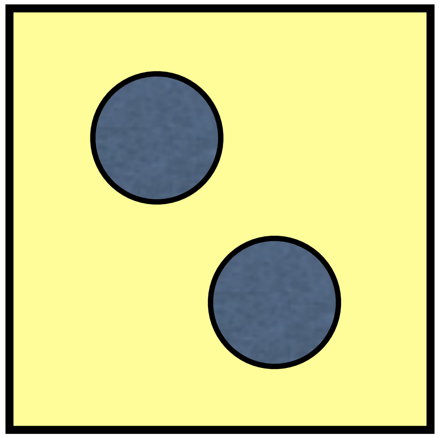
(RBC and UKQCD Collaborations)

History & status of finite-volume formalism for 3 particles

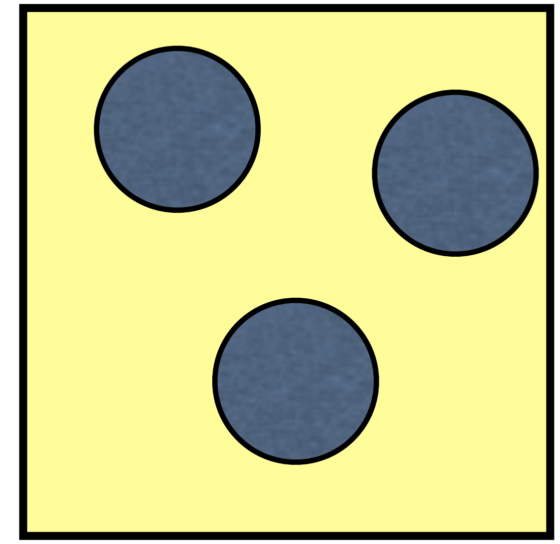
Sketch of history for three particles

- [Beane, Detmold, Savage et al. 07-11] studied ground state energies of $N\pi^+, MK^+, N\pi^+ + MK^+$ systems, and determined 3-particle interactions for particles at rest
- [Polejaeva & Rusetsky 12] showed in NREFT that 3 body spectrum determined by $2 \to 2$ & $3 \to 3$ infinite-volume scattering amplitudes
- [Hansen & SRS 14, 15] derived quantization condition (QC3) for 3 identical scalars in generic, relativistic EFT, working to all orders in Feynman-diagram expansion, keeping all angular momenta—"RFT approach"
- [Hammer & Rusetsky 17] derived QC3 using NREFT—greatly simplified derivation
- [Mai & Döring 17] obtained QC3 using unitary, relativistic representation of $3 \to 3$ amplitude—"FVU approach"
- [Blanton & SRS 20] showed equivalence of RFT & FVU approaches
- [Hansen, Romero-López, SRS 21] derived formalism for determining $K \to 3\pi$ amplitude

Additional issues with 3 particles



vs



e.g. 2π with $E_{\rm CM} < 4M_\pi$

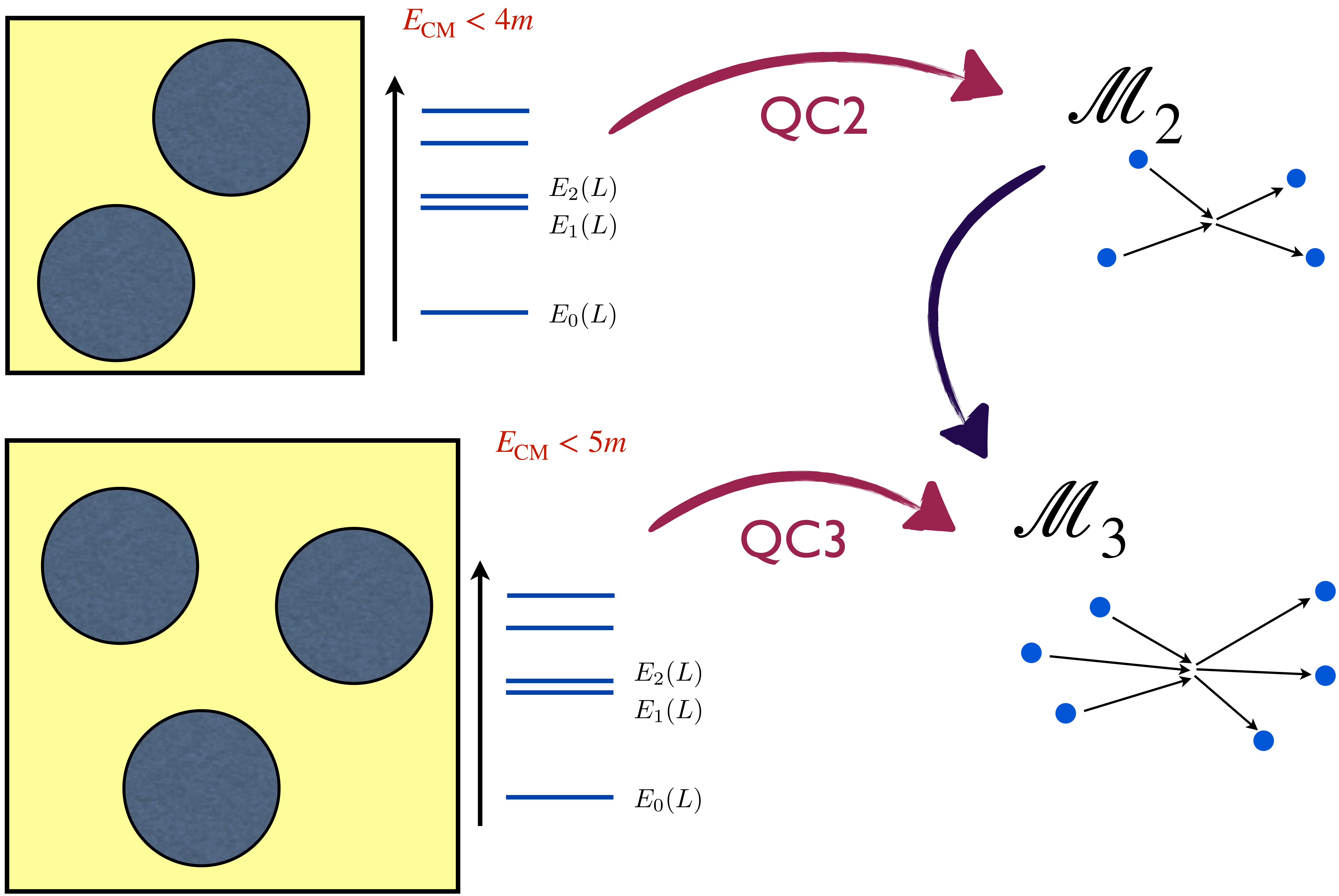
e.g. 3π with $E_{\rm CM} < 5M_\pi$

(For simplicity, assume G-parity-like Z_2 symmetry, so no $2 \leftrightarrow 3$ transitions; formalism can be generalized)

- Energy shifts $\Delta E_n = E_n - E_{n,\rm free} \sim 1/L^3$
- Scattering amplitude in each partial wave, at given $E_{\rm CM}$, is a (complex) number

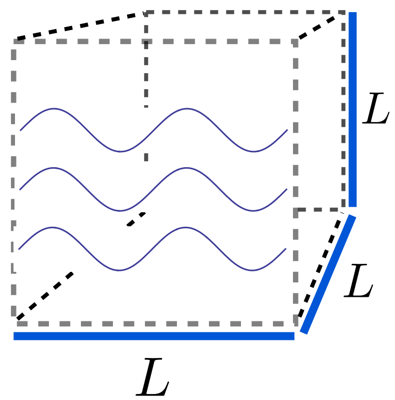
- Dominant contribution from pairwise interactions, $\Delta E_n \sim 1/L^3$
- 3-particle interactions give subleading contributions $\propto 1/L^6$
- Scattering amplitude $\mathscr{M}_3$ at given $E_{\rm CM}$, is a (complex) function of Dalitz-plot variables, and incorporates final-state interactions
- $\mathscr{M}_3$ has divergences for physical momenta

Structure of the result (Z_2 symmetry)



Two-step method

2 & 3 particle
Spectra from LQCD



Infinite-volume K matrix:
Obtained from Feynman diagrams
using PV prescription for poles;
Real, free of unitary cuts

Quantization conditions

QC2: $\det\left[F^{-1} + \mathcal{K}_2\right] = 0$

QC3: $\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right] = 0$

[These are the RFT forms, and assume $\mathbb{Z}_2$ symmetry]

Intermediate infinite-volume K matrix:
A short-distance, real, three-particle
interaction free of unitary cuts, and
with physical divergences subtracted;
unphysical since depends on cutoff

Integral equations in infinite volume

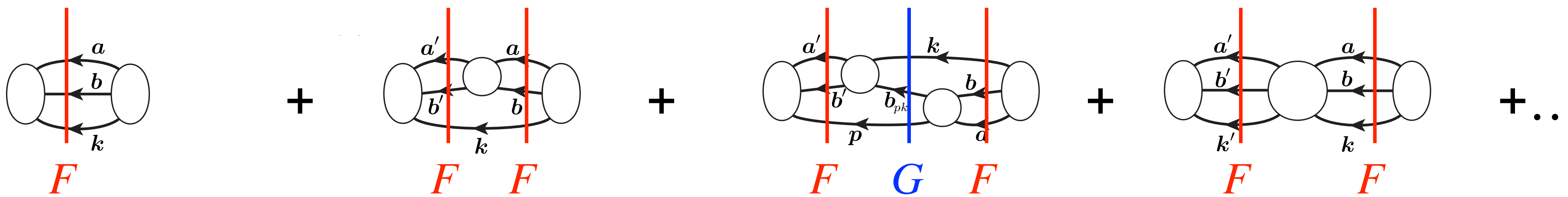
Incorporates initial- and
final-state interactions

Scattering amplitude $\mathcal{M}_3$

Further details of QC3

$$\det\left[F_3^{-1} + \mathscr{K}_{\mathrm{df},3}\right] = 0$$

- Derived by determining power-law volume dependence of finite-volume 3-particle correlation functions to all orders in a skeleton expansion in a generic relativistic EFT



  - Volume dependence arises from 3-particle cuts

- F_3 contains two-particle interactions ($\mathscr{K}_2$) and kinematic functions (F & G)

$$F_3 = \frac{1}{2\omega L^3}\left[\frac{F}{3} - F\frac{1}{\mathscr{K}_2^{-1} + F + G}F\right]$$

Status: formalism

[Full references at end of slides]

- 3 identical spinless particles [Hansen, SRS; Hammer, Pang, Rusetsky; Mai, Döring]
- Mixing of two- and three-particle channels for identical spinless particles [Briceño, Hansen, SRS]
- 3 degenerate but distinguishable particles, e.g 3π with isospin 0, 1, 2, 3 [Hansen, Romero-López, SRS]
- 3 nondegenerate particles, e.g. $D_s^+ D^0 \pi^-$ [Blanton, SRS]
- (Single-channel) 2+1 systems, e.g. $\pi^+\pi^+K^+$ [Blanton, SRS]
- 3 identical spin-½ particles, e.g. 3 neutrons [Draper, Hansen, Romero-López, SRS]

Many resonances can now be studied!

Resonance	$I_{\pi\pi\pi}$	J^P	Decays
$\omega(782)$	0	1^-	$\pi^+\pi^0\pi^-$
$h_1(1170)$	0	1^+	$\rho\pi \to 3\pi$
$\omega_3(1670)$	0	3^-	3π, 5π
$\pi(1300)$	1	0^-	$\rho\pi \to 3\pi$
$a_1(1260)$	1	1^+	3π, $K\bar{K}\pi$
$\pi_1(1400)$	1	1^-	$\eta\pi$, 3π?
$\pi_2(1670)$	1	2^-	3π, $K\bar{K}\pi$
$a_2(1320)$	1	2^+	3π, $K\bar{K}$, 5π, $\eta\pi$
$a_4(1970)$	1	4^+	3π, $K\bar{K}$, 5π, $\eta\pi$

Status: applications

[References at end of slides]

- $3\pi^+$: determined parameters in threshold expansion of $\mathscr{K}_{\mathrm{df},3}$, including pair interactions in s- and d-waves; integral equations solved for s-wave interactions only
- $3K^+$: determined s- and d-wave parameters in $\mathscr{K}_{\mathrm{df},3}$
- ϕ^4: extracted $\mathscr{K}_{\mathrm{df},3}$ in single-scalar theory; extracted 3-particle resonance parameters in two-scalar theory, using RFT and FVU approaches
- 3π with $I=1$: first study of $a_1(1260)$ with formalism based on 2 levels; solved integral equations in FVU approach
- $\pi^+\pi^+K^+$ & $K^+K^+\pi^+$: determined s- and p-wave parameters in $\mathscr{K}_{\mathrm{df},3}$; found evidence for small discretization effects
- Integral equations solved for complex energies for simple system with near-unitary two-particle interactions and Efimov states (bound or resonant)
- ChPT: LO results for $3\pi^+$, $\pi^+\pi^+K^+$, $K^+K^+\pi^+$, $3K^+$, including a^2 effects: agree in rough magnitude but not in detail with results from LQCD calculations
- ChPT: NLO result for $3\pi^+$; greatly improves agreement with LQCD results

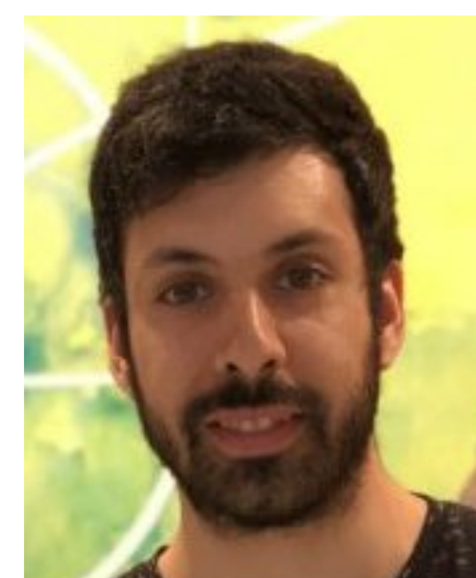
$\pi^+\pi^+K^+$ and $K^+K^+\pi^+$ amplitudes using LQCD

[Draper, Hanlon, Hörz, Morningstar, Romero-López & SRS, 2302.13587 (JHEP)]

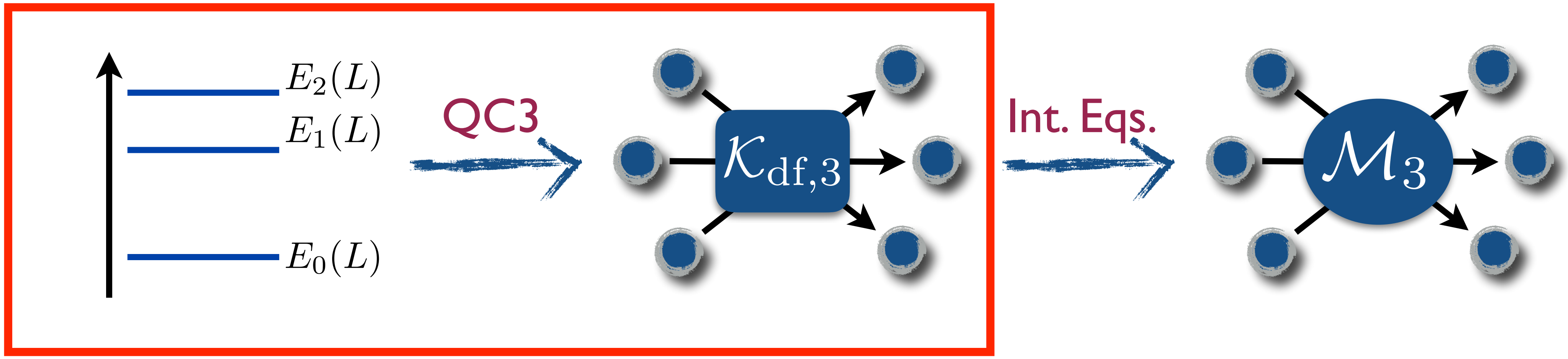
A step on the way to $T_{cc} \to DD\pi$, etc.



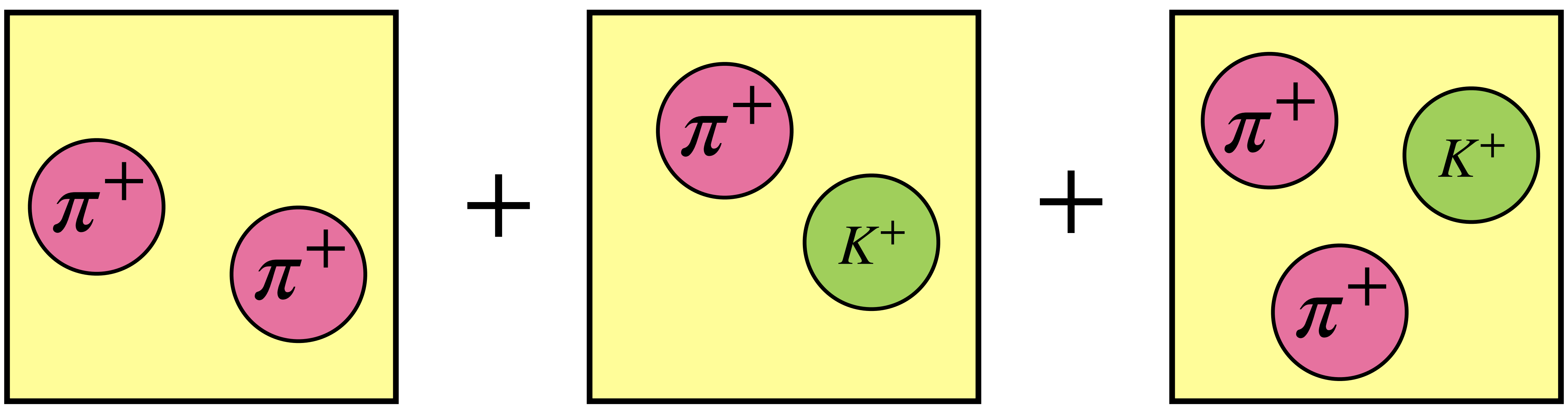




Strategy



- Consider multiparticle system with weakly repulsive interactions—pions and kaons at maximal isospin ($2\pi^+/3\pi^+, 2K^+/3K^+, 2\pi^+/\pi^+K^+/3K^+, 2K^+/\pi^+K^+/3K^+$)
  - No resonances in two-particle subchannels or in three-particle system
  - Simultaneously fit to several spectra,; for example, to obtain the $\pi^+\pi^+K^+$ interaction need:



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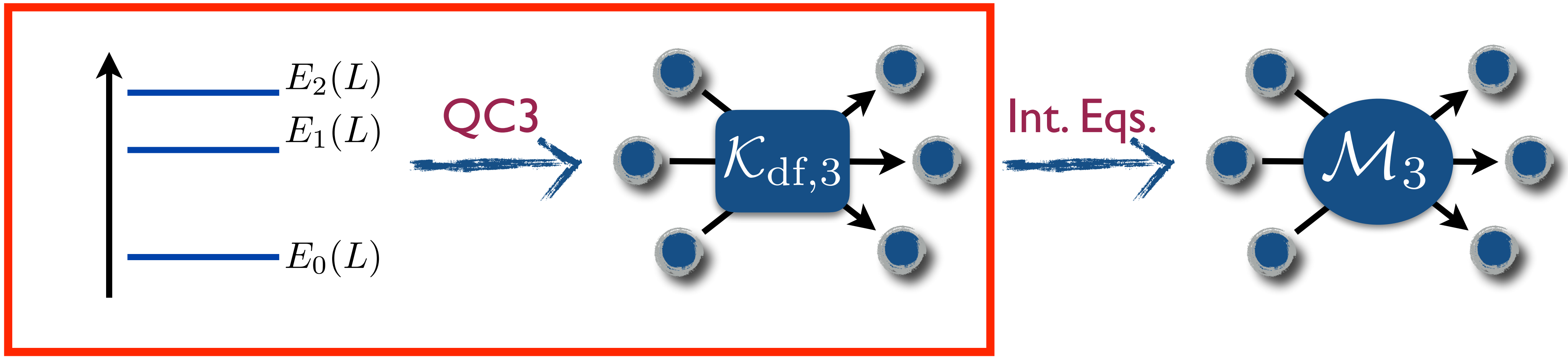

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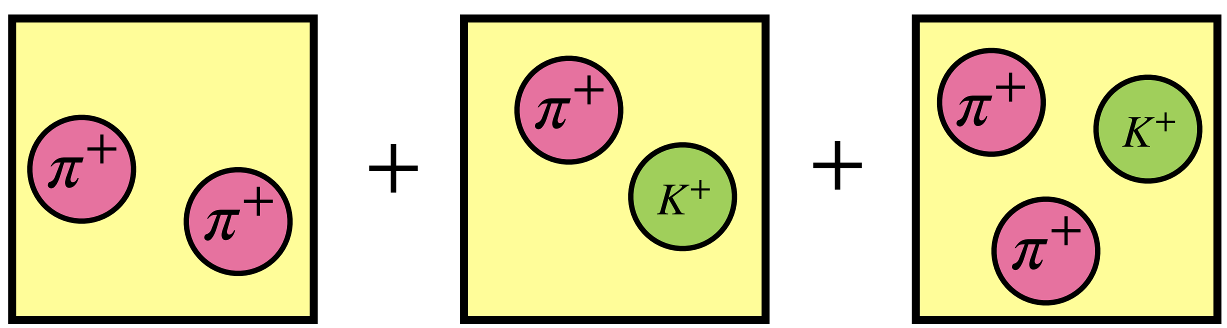


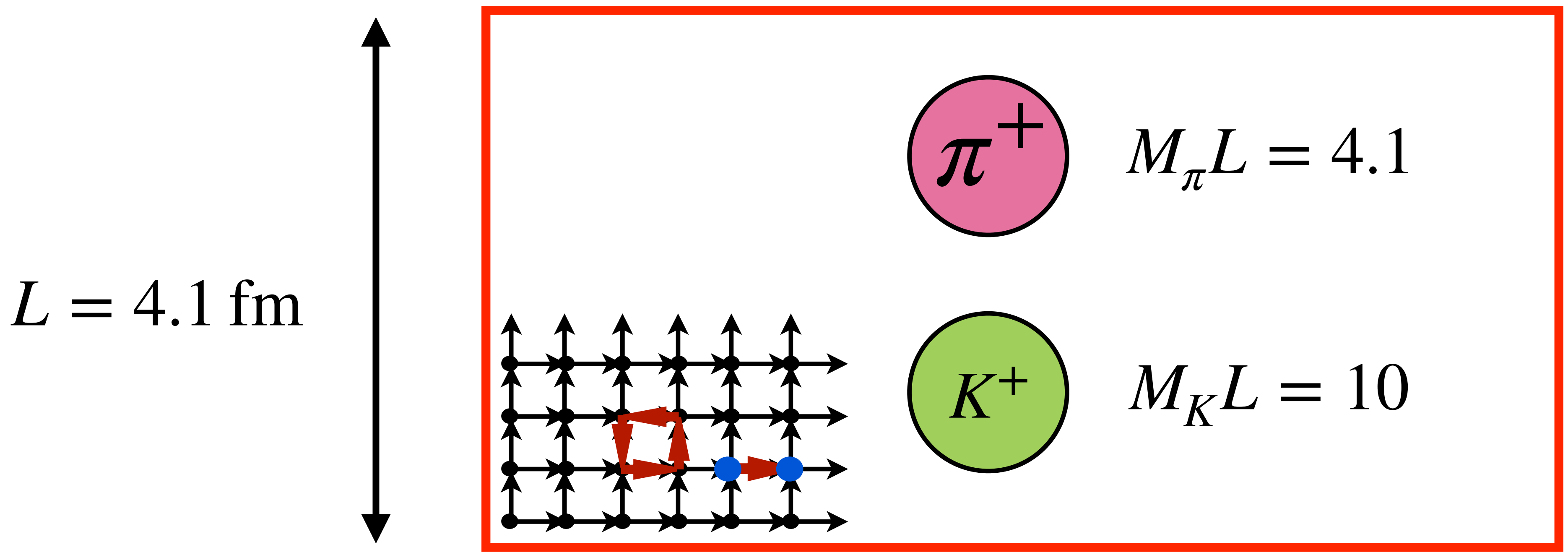
- Parametrize $\mathcal{K}_{\mathrm{df},3}$ (and $\mathcal{K}_2$) as the most general smooth function consistent with particle interchange, time-reversal and parity symmetries, using an expansion about threshold
  - Generalization of the effective-range expansion for $\mathcal{K}_2$; here keep first two terms
  - s-wave interactions in $\pi^+\pi^+$ (sub)channel, s- and p-wave in π^+K^+; 9 or 10 parameters in all

Lattices used in pilot calculation

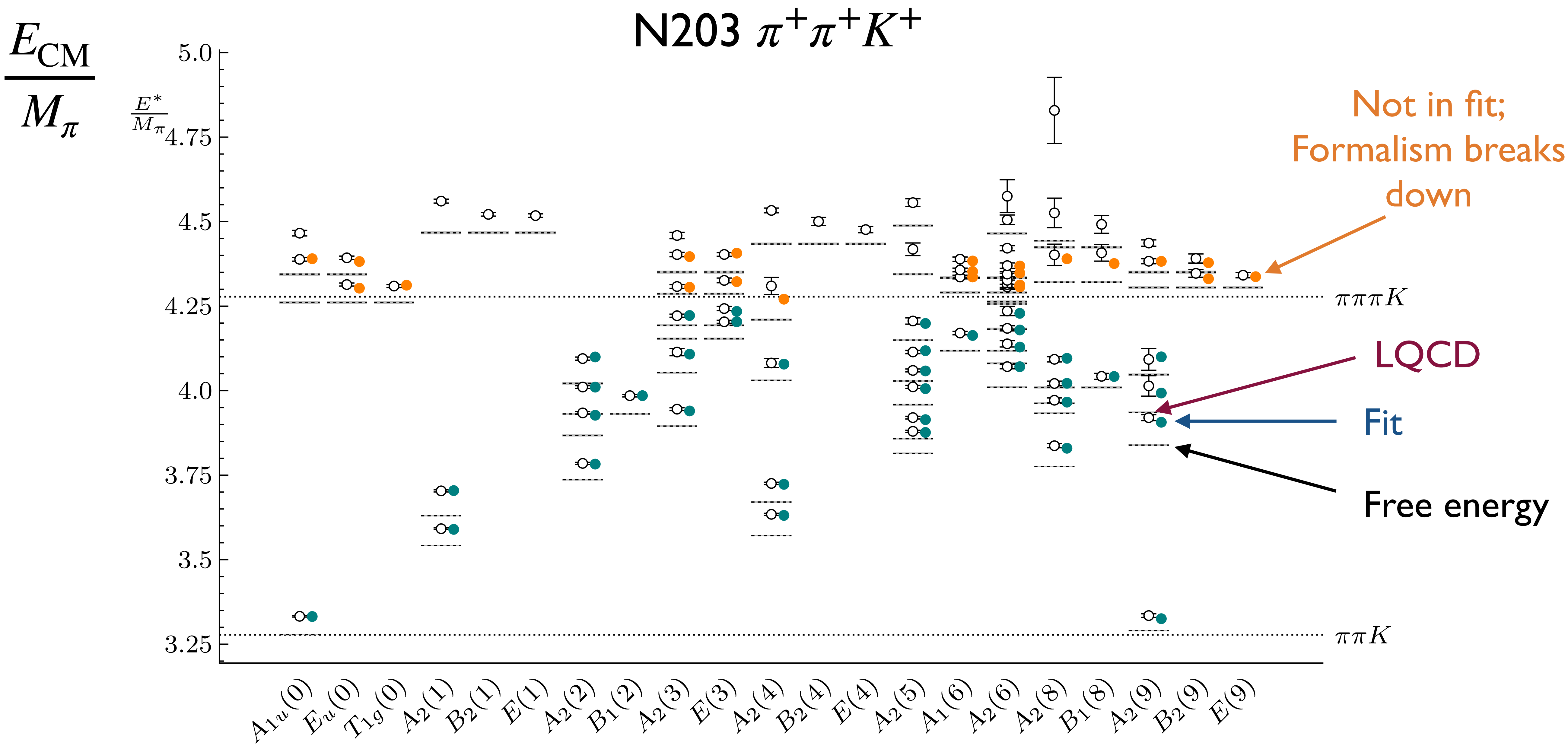
- Improved Wilson fermions at $a = 0.064\,\mathrm{fm}$ (CLS lattices)

	$(L/a)^3 \times (T/a)$	M_π [MeV]	M_K [MeV]	$N_{\rm cfg}$	$t_{\rm src}/a$	$N_{\rm ev}$	dilution	$N_r(\ell/s)$
N203	$48^3 \times 128$	340	440	771	32, 52	192	(LI12,SF)	6/3
D200	$64^3 \times 128$	200	480	2000	35, 92	448	(LI16,SF)	6/3

D200 configurations



Example of fit

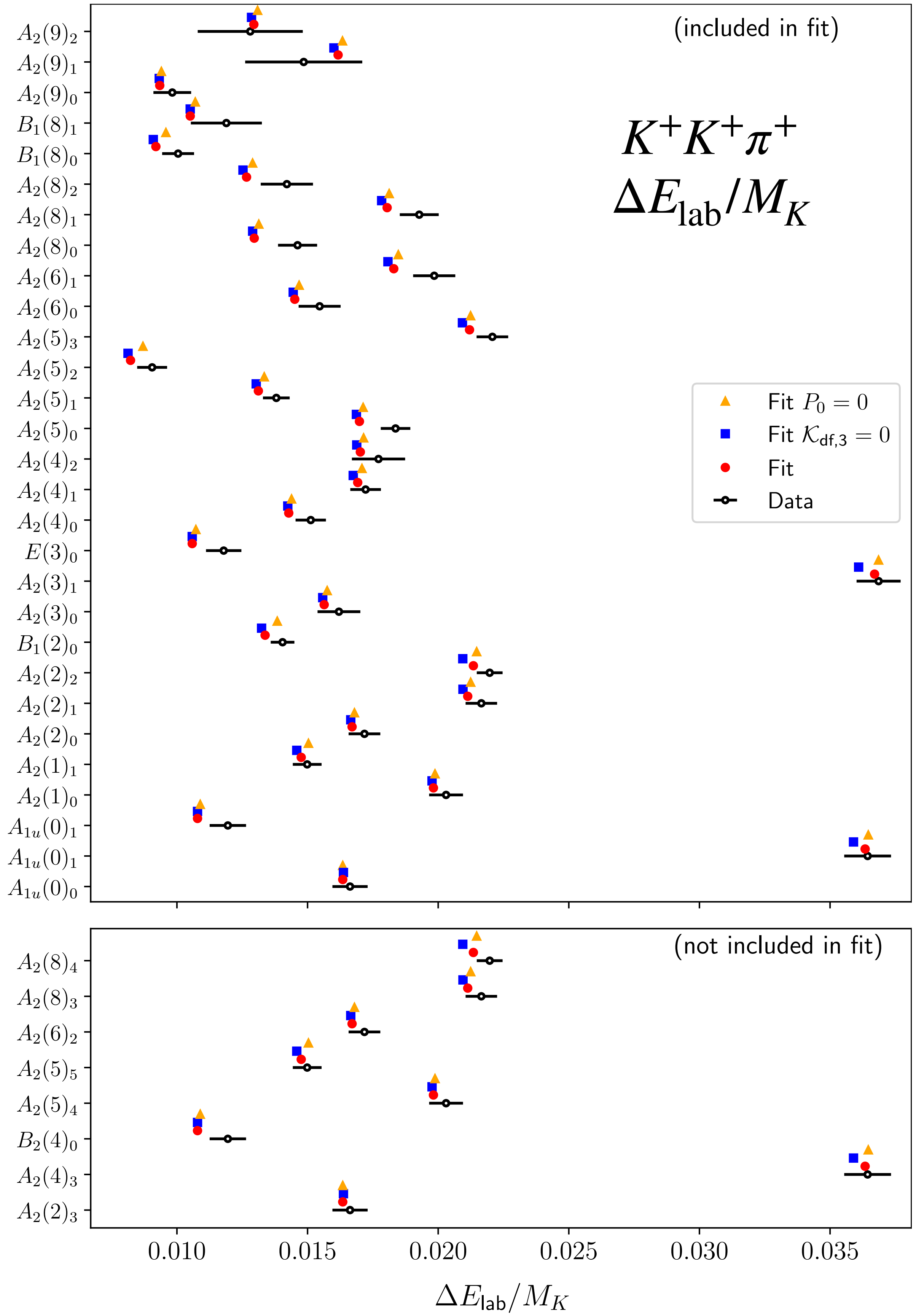


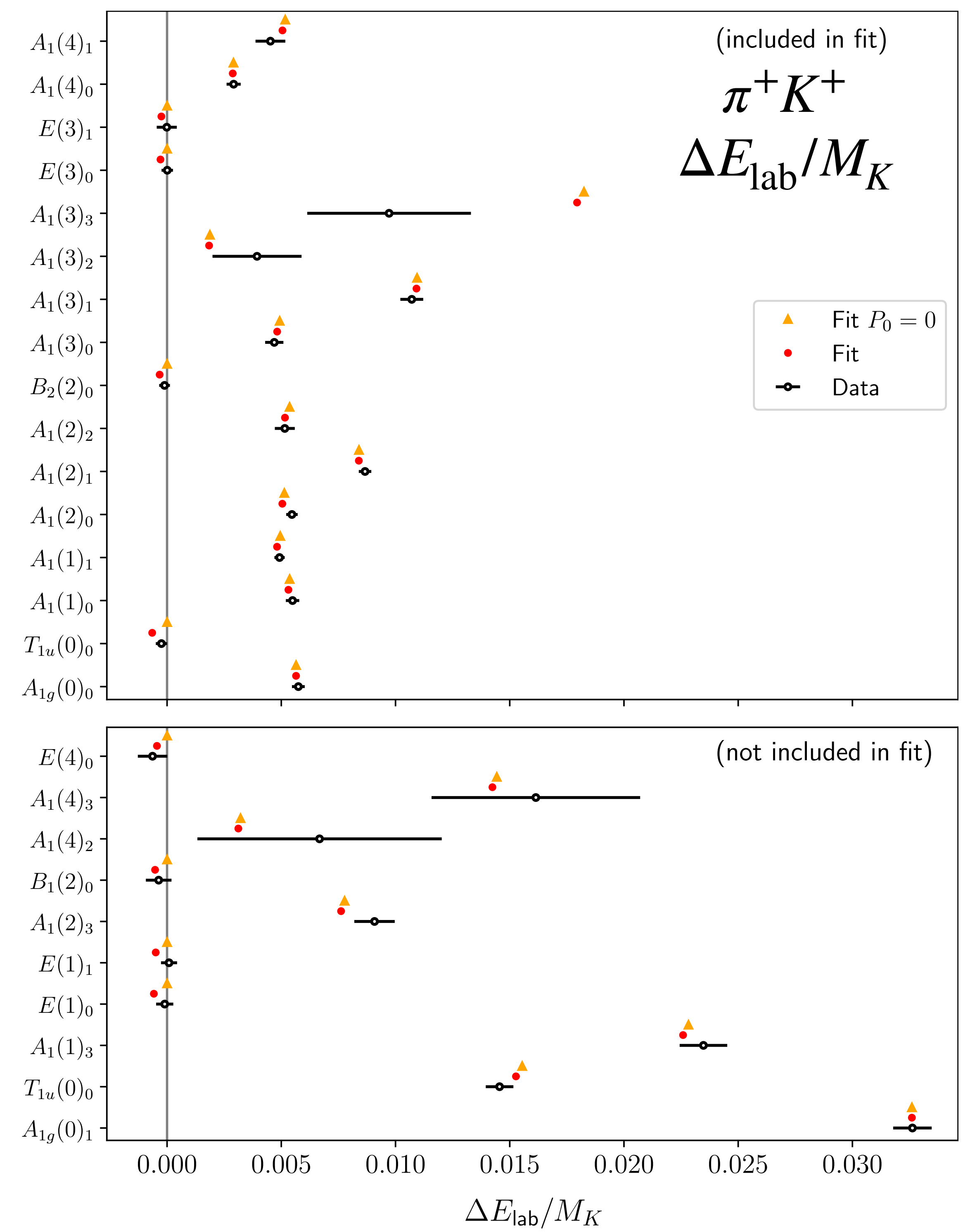
Simultaneous fit to 27 $\pi^+\pi^+$, 19 π^+K^+, & 36 $\pi^+\pi^+K^+$ levels with 9 parameters

$\chi^2/\mathrm{DOF} = 119/(82-9)$

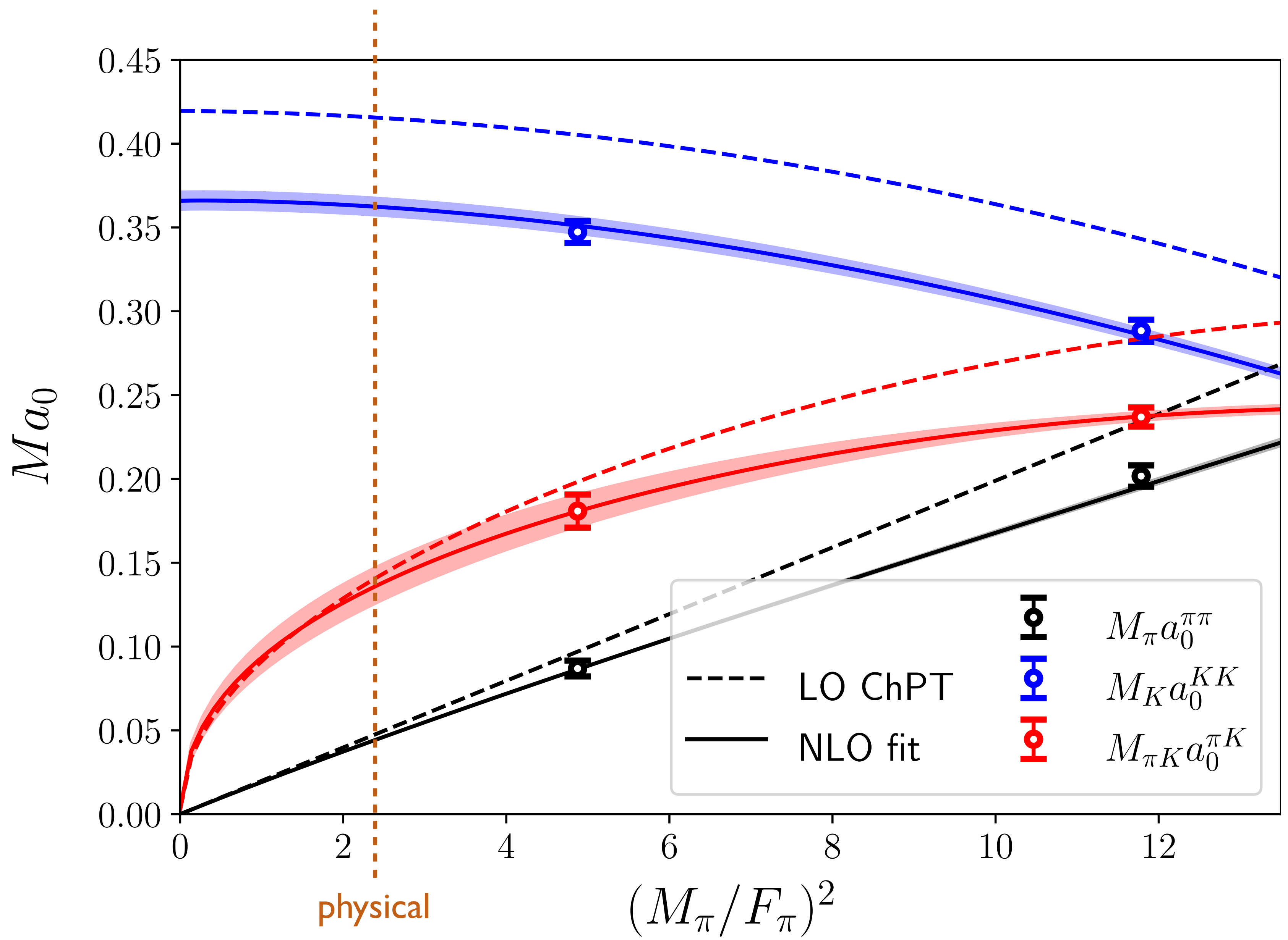
Fit is to lab-frame shifts

Simultaneous fit to 28 K^+K^+, 16 π^+K^+, & 29 $K^+K^+\pi^+$ levels with 10 parameters on D200: $\chi^2/\mathrm{DOF} = 162/(73-10)$



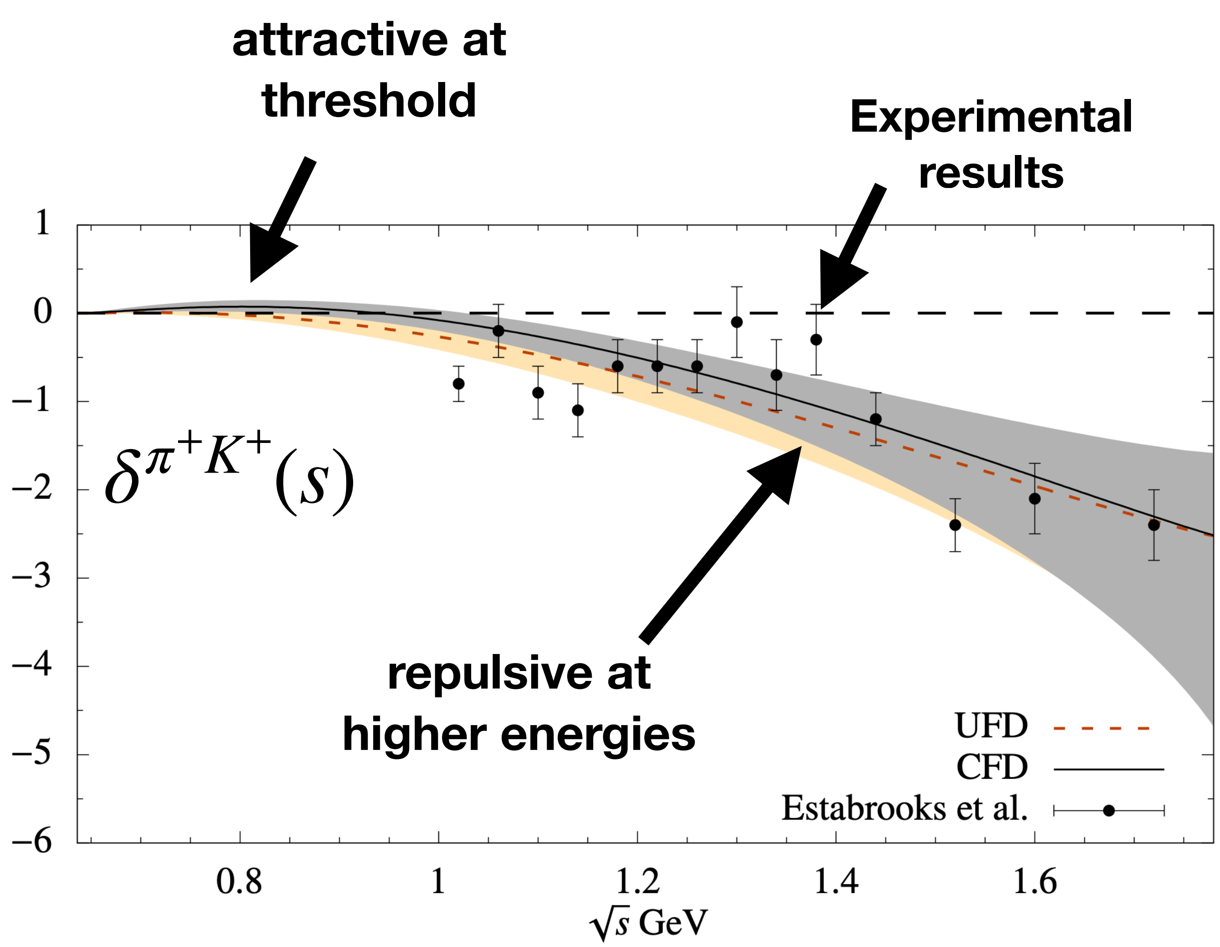


Results: scattering lengths

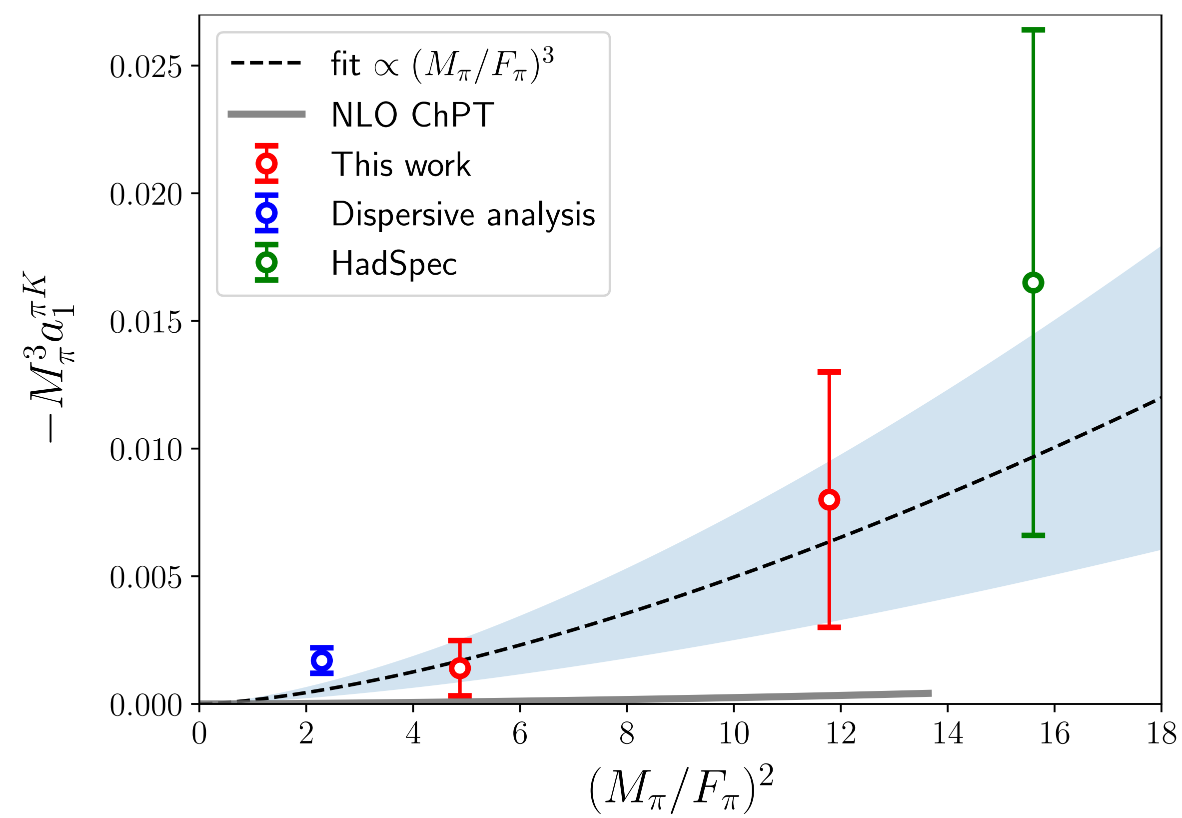


- 2-particle s-wave scattering lengths are well determined
- All are repulsive and consistent with ChPT
  - Evidence for small discretization errors

P-wave π^+K^+ scatt. Length

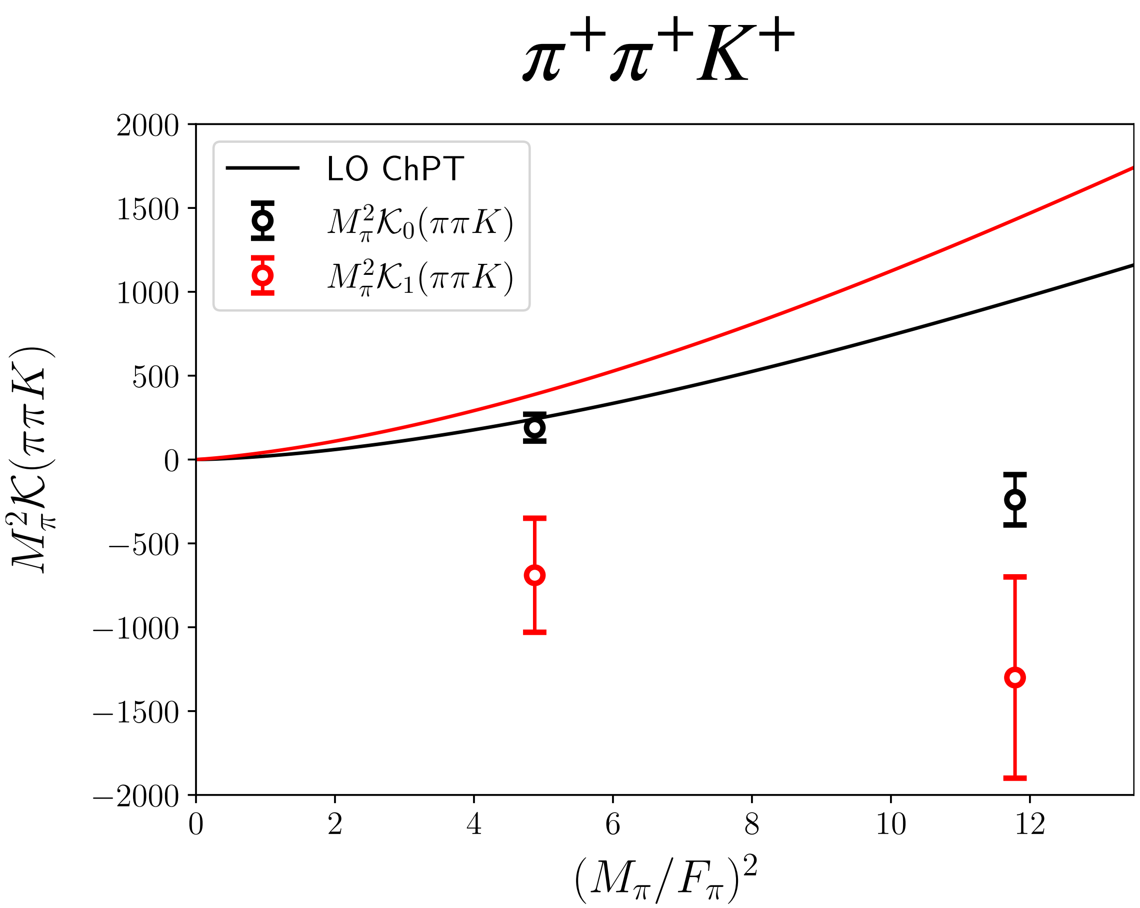


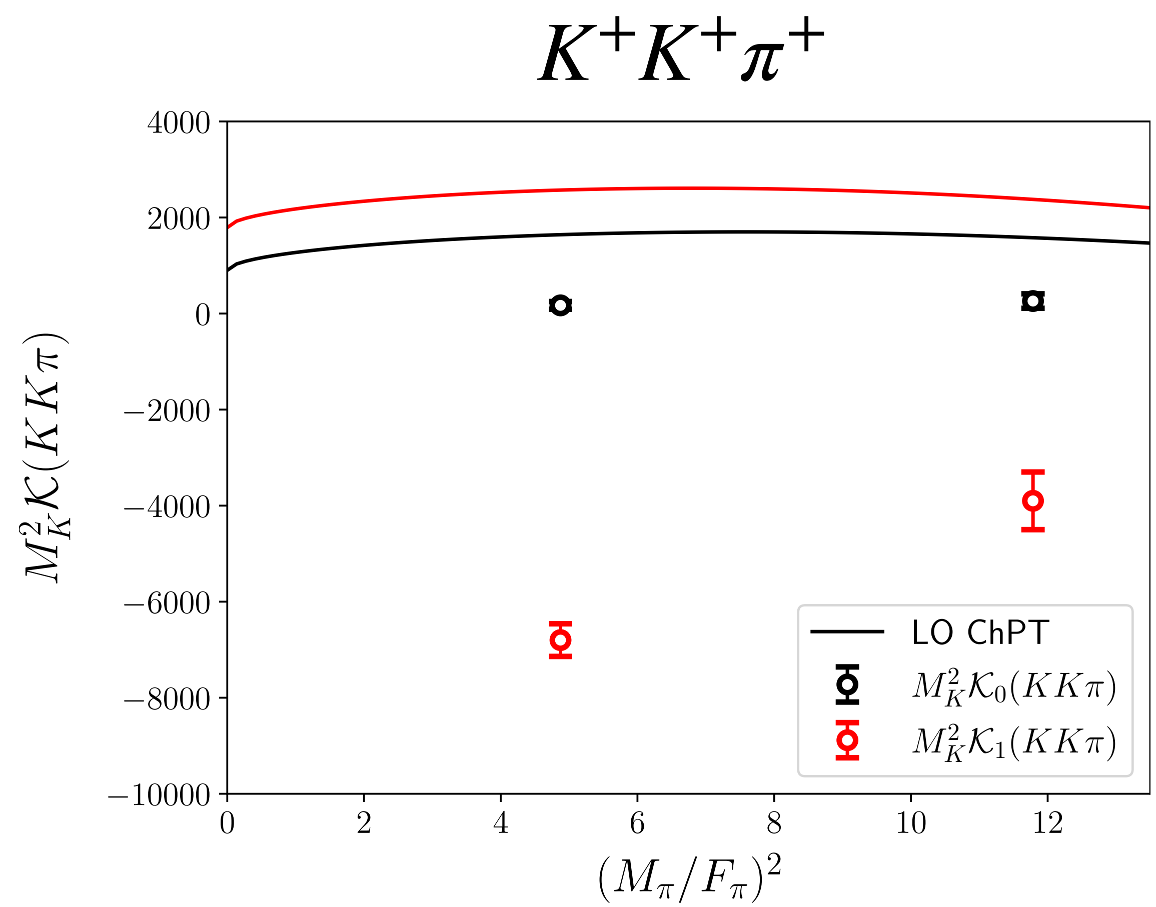
[Pelaez, Rodas, 2010.11222]



- Find evidence for **attractive** p-wave scattering length
  - Consistent with dispersive analysis

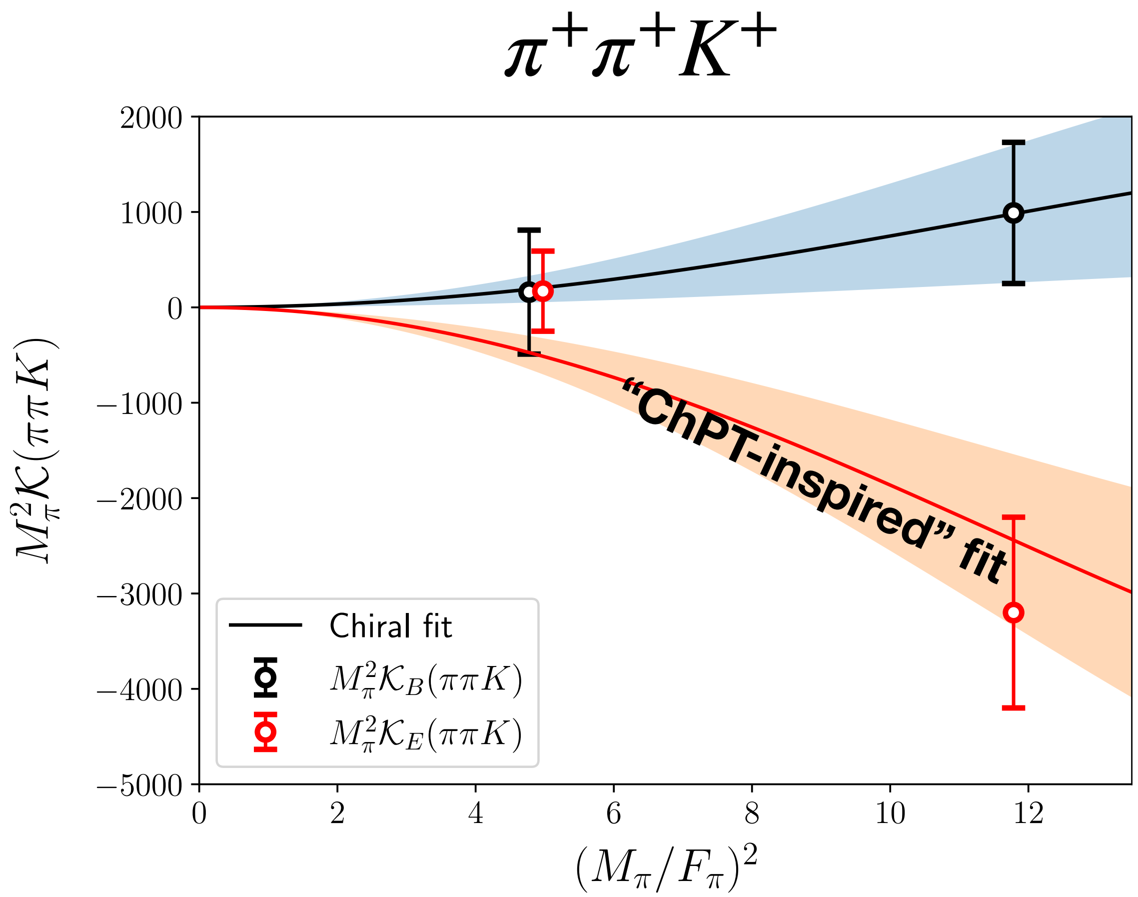
s-wave contributions to $\mathcal{K}_{\mathrm{df},3}$

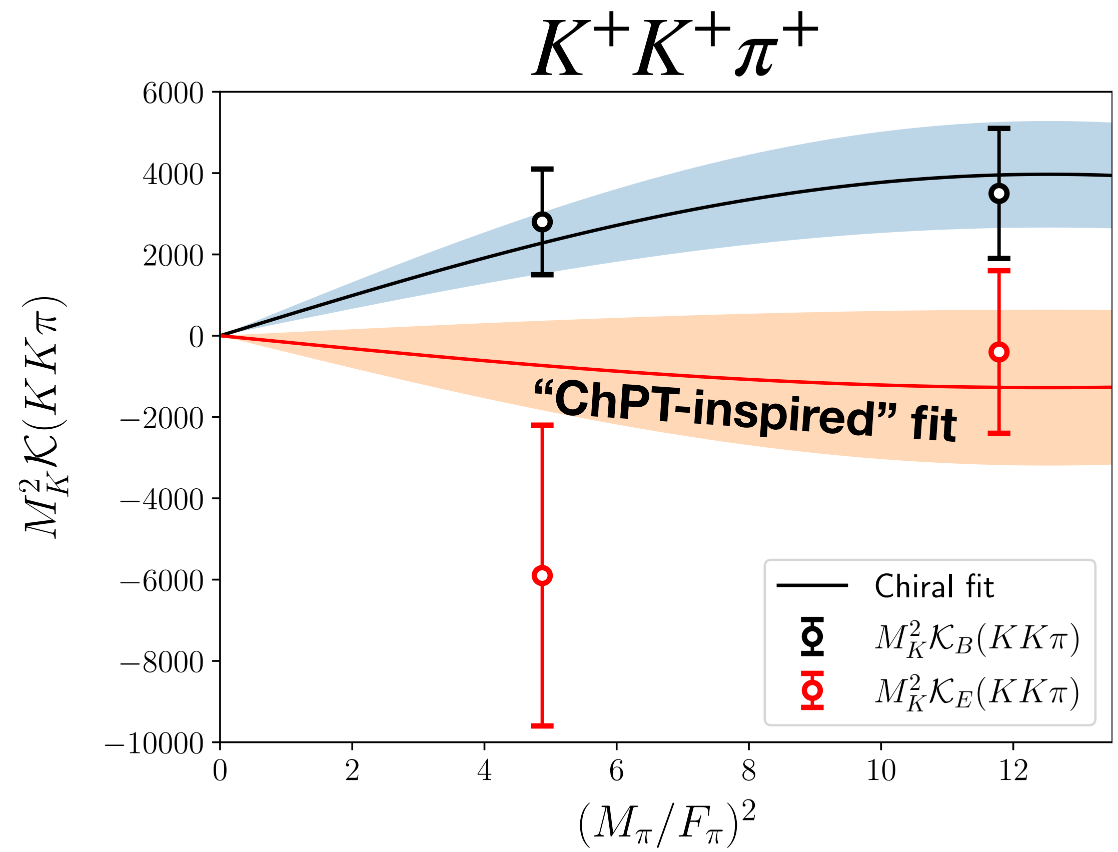




- Evidence for nonzero values ($2-5\sigma$)
- Overall effect of $\mathcal{K}_{\mathrm{df},3}$ is repulsive
- LO ChPT predicts opposite sign (but see later)

p-wave contributions to $\mathcal{K}_{\mathrm{df},3}$

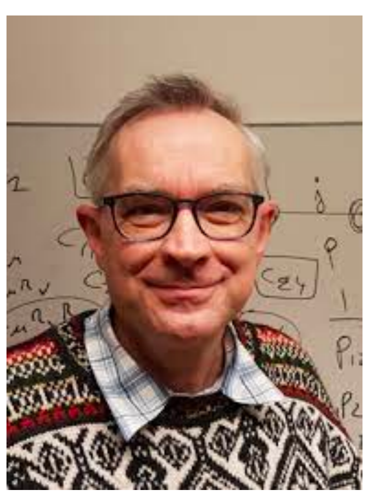


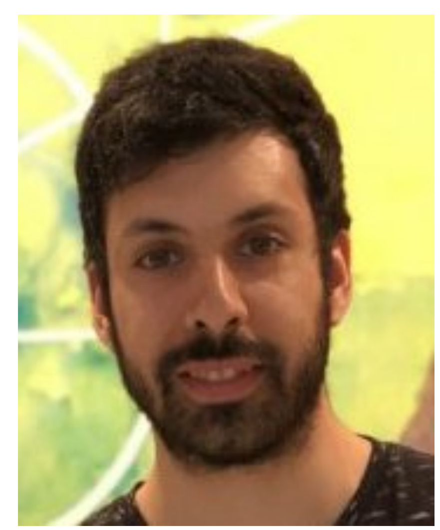
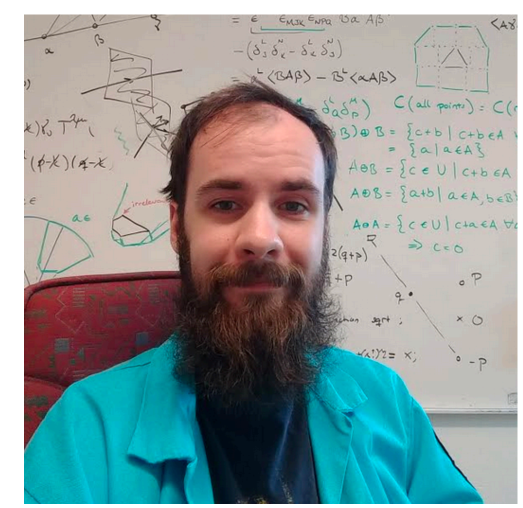


- Evidence for nonzero values in some cases
  - $\mathcal{K}_E$ is only contribution of $\mathcal{K}_{\mathrm{df},3}$ to nontrivial irreps
- Appear at NLO in ChPT—prediction not yet available

NLO ChPT results for $\mathcal{K}_{\mathrm{df},3}$ for $3\pi^+ \to 3\pi^+$

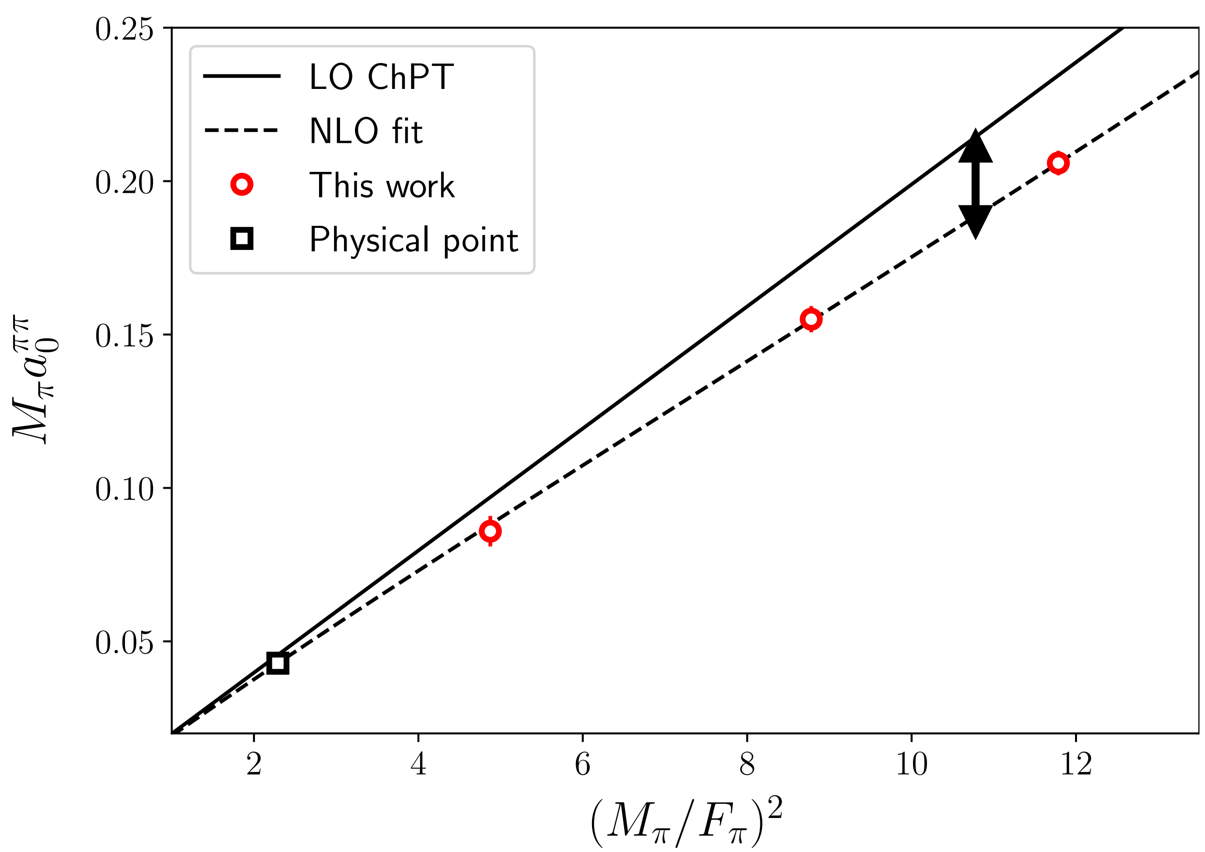
[Baeza-Ballesteros, Bijnens, Husek, Romero-López, SRS, Sjö, 2303.13206]

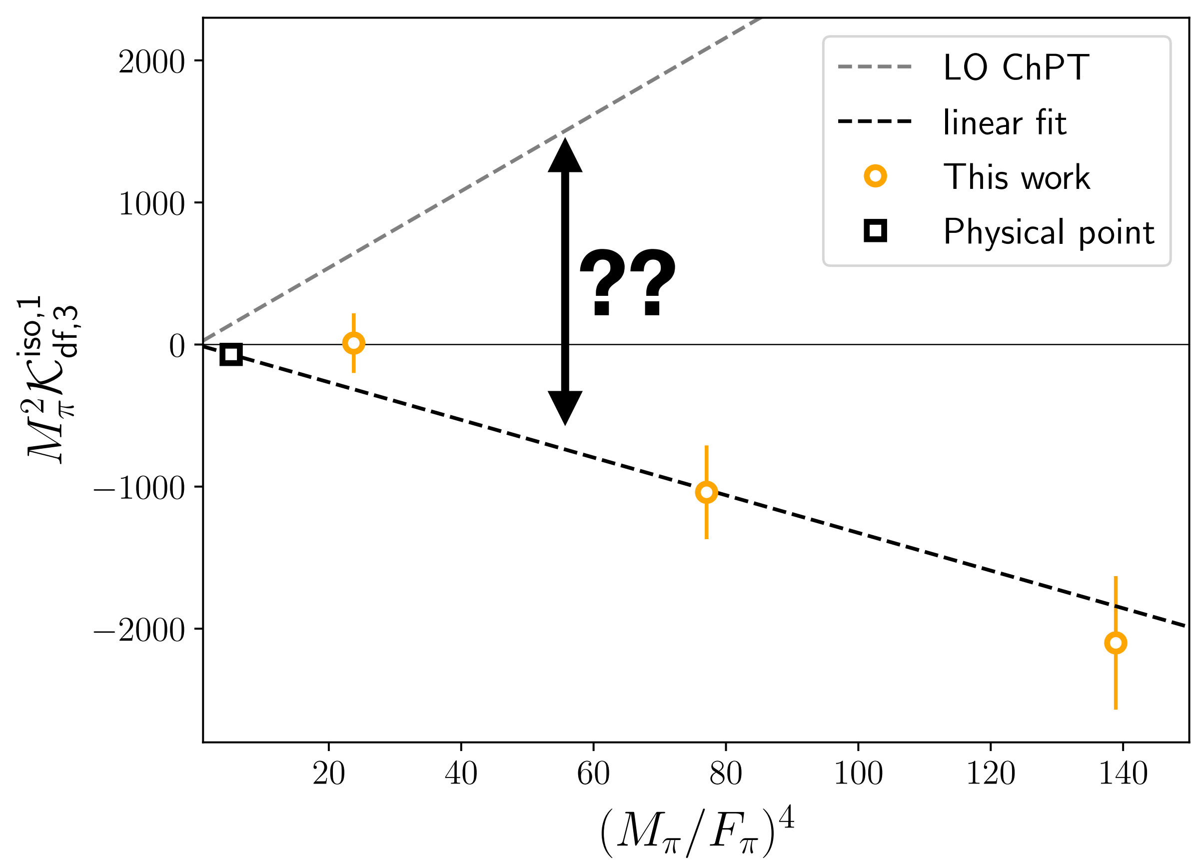



$2\pi/3\pi$ K matrices vs ChPT

$2\pi^+$ scattering length



$3\pi^+$ K matrix

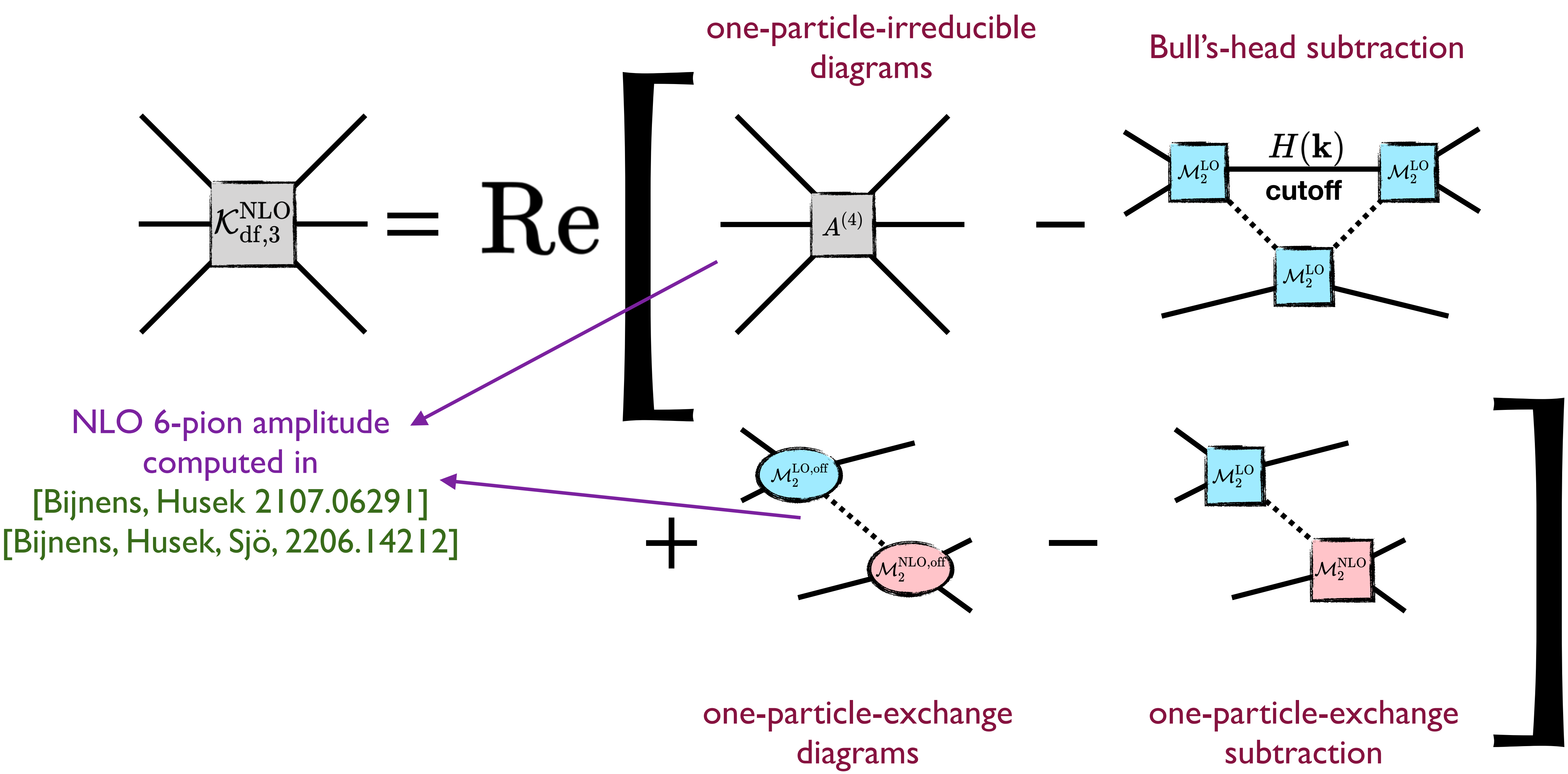


[Results from Blanton, Hanlon, Hörz, Morningstar, Romero-López, SRS, 2106.05590 (JHEP)]

- LO ChPT describes 2-pion sector well
- Large discrepancy in 3-pion sector!

NLO ChPT for $\mathcal{K}_{\mathrm{df},3}$

- Integral equations simplify to: $\boxed{\mathcal{K}_{\mathrm{df},3}^{\mathrm{NLO}} = \mathrm{Re}\, \mathcal{M}_{\mathrm{df},3}^{\mathrm{NLO}}}$



Threshold expansion for $\mathscr{K}_{\mathrm{df},3}$

- $\mathscr{K}_{\mathrm{df},3}$ is a real, smooth function which is Lorentz, P and T invariant
- Expand about threshold in powers of $\Delta = (s-9M_\pi^2)/9M_\pi^2,\ \tilde{t}_{ij} = (p_i' - p_j)^2/9M_\pi^2, \ldots$

Depend on CM energy

Angular dependence

$$\mathcal{K}_{\mathrm{df},3} = \boxed{\mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,2}\Delta^2} + \boxed{\mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B} + \mathscr{O}(\Delta^3)$$

$$\boxed{\Delta_B = \sum_{i,j=1}^{3} \tilde{t}_{ij}^2 - \Delta^2,}$$

- Can separate terms in fit based on dependence on energy and rotational properties
  - E.g. only $\mathscr{K}_B$ contributes to nontrivial irreps

NLO ChPT results for $\mathscr{K}_{\rm df,3}$

$\kappa = 1/(16\pi^2)$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35+12\log 3) - \mathcal{D}_0 + 111L + \ell^{\rm r}_{(0)}\right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}\left(1999+1920\log 3\right) - \mathcal{D}_1 + 384L + \ell^{\rm r}_{(1)}\right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923-420\log 3) - \mathcal{D}_2 + 360L + \ell^{\rm r}_{(2)}\right],$$

$$\mathcal{K}_{\rm A} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809-1050\log 3) - \mathcal{D}_{\rm A} - 9L + \ell^{\rm r}_{({\rm A})}\right],$$

$$\mathcal{K}_{\rm B} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698-245\log 3) - \mathcal{D}_{\rm B} + 54L + \ell^{\rm r}_{({\rm B})}\right].$$

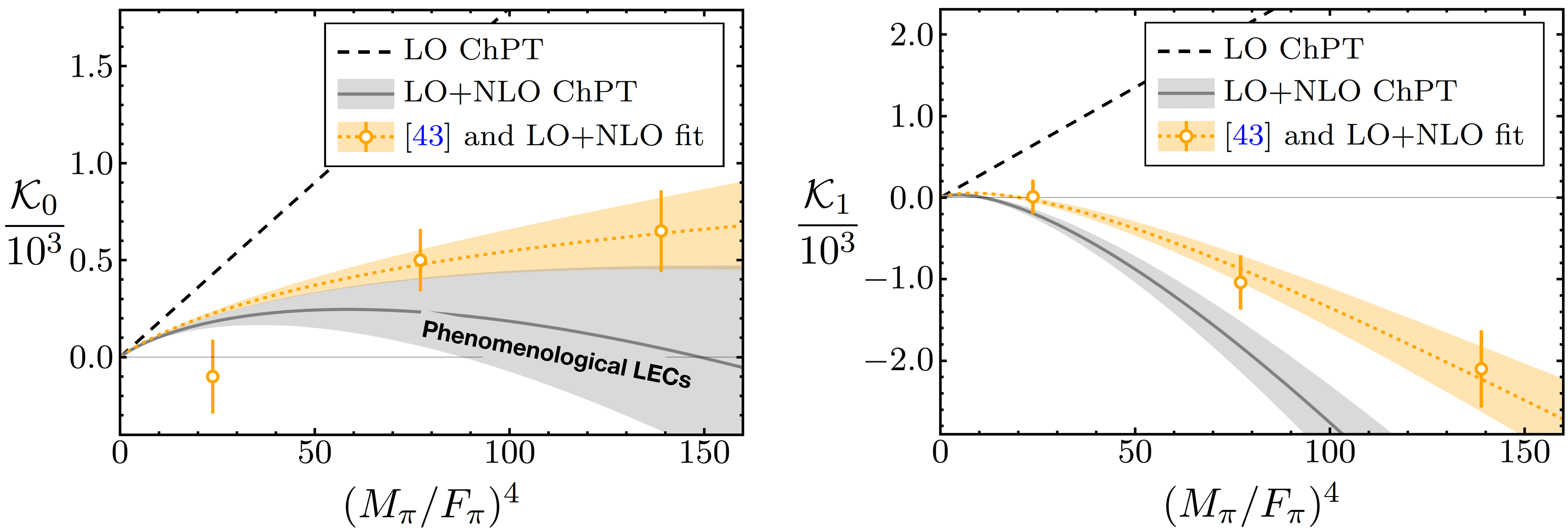
$L \equiv \kappa \log(M_\pi^2/\mu^2)$

LECs

Numerical coefficients
Depend on cutoff H(**k**)

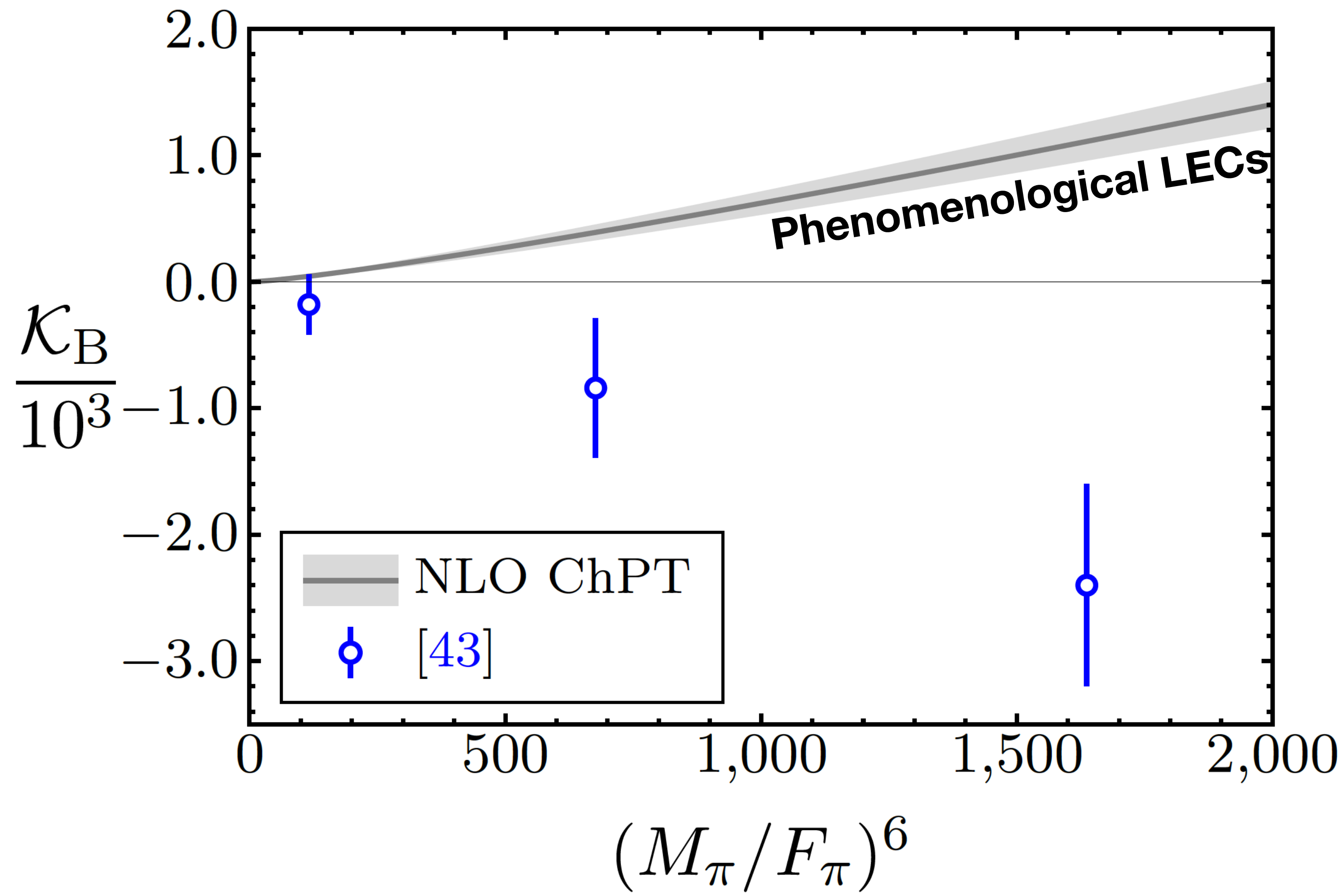
μ-dependence cancels

Comparison to LQCD



- (Very) large NLO corrections
  - Discrepancy with LO ChPT resolved!
  - ChPT not trustworthy for $\mathscr{K}_1$

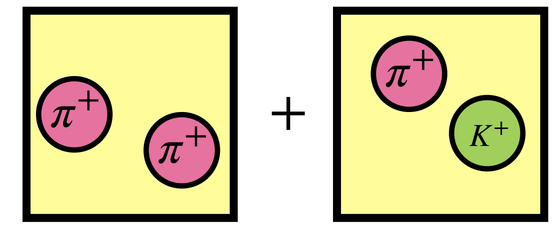
Comparison to LQCD



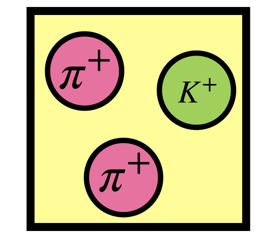
- $\mathscr{K}_B$ first appears at NLO in ChPT
- Discrepancy may be resolved by NNLO terms?

Summary & Outlook

Summary



- Two-particle sector is entering precision phase
  - Frontier is two nucleons, which are more challenging for LQCD
- Major steps have been taken in the three-particle sector



  - Formalism well established & cross checked, and almost complete
  - Several applications to three-particle spectra from LQCD
  - Initial discrepancy with LO ChPT explained by large NLO contributions
  - Integral equations solved in several cases
  - Path to a calculation of $K \to 3\pi$ decay amplitudes is now open

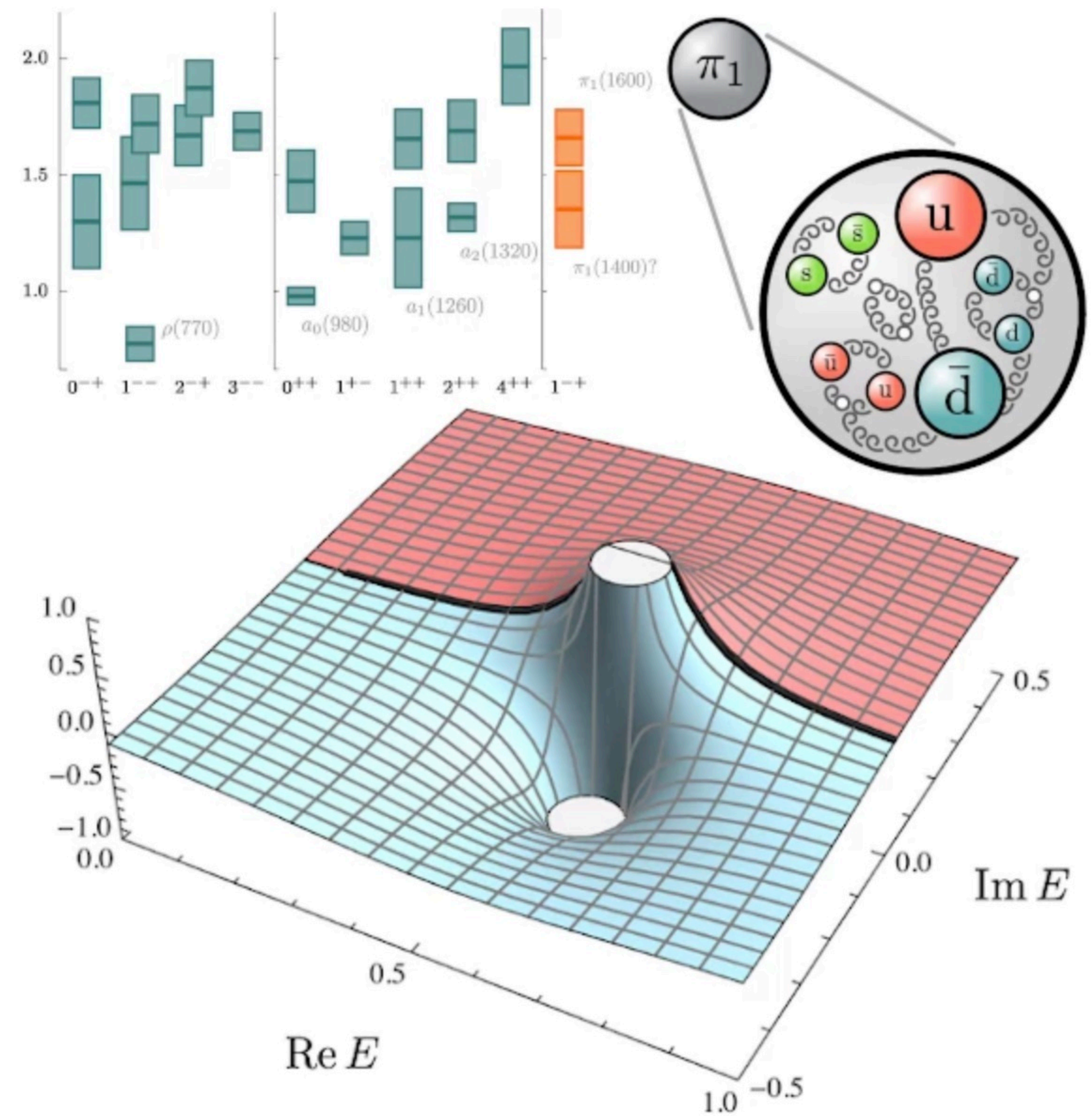
Outlook

- Generalize formalism to broaden applications
  - 3 nucleons with $I = \frac{1}{2}$ (nnp & ppn)
  - $T_{cc}(3875, I=0, J^P = 1^+?) \to D^0 D^0 \pi^+,\ D^+ D^0 \pi^0,\ D^+ D^+ \pi^-$
  - Accessing the WZW term: $K\bar{K} \leftrightarrow \pi^+\pi^0\pi^-(I=0)$
  - $N(1440, J^P = \frac{1}{2}^+) \to N\pi,\ N\pi\pi$
  - $J^{PC}, I^G = 1^{-+}, 1^-:\ \ \pi_1(1600) \to \eta\pi,\ 3\pi,\ KK\pi\pi,\ \eta\pi\pi\pi,\ 5\pi$
- Extend implementations using LQCD simulations
  - $3\pi^+,\ 3K^+,\ \pi^+\pi^+K^+,\ K^+K^+\pi^+$ at physical quark masses
  - I=0,1 three-particle resonances ($\omega,\ a_1, \ldots$)
- Extend applications of integral equations in the presence of three-particle resonances, e.g. T_{cc}
- Move on to 4 particles!

ExoHad collaboration

ExoHad Collaboration exohad.org People Events Talks Publications

EXOHAD
EXOTIC HADRONS TOPICAL COLLABORATION



The Exo(tic) Had(ron) Collaboration started in 2023 to explore all aspects of exotic hadron physics, from predictions within lattice QCD, through reliable extraction of their existence and properties from experimental data, to descriptions of their structure within phenomenological models.

Thank you!
Questions?

References

RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

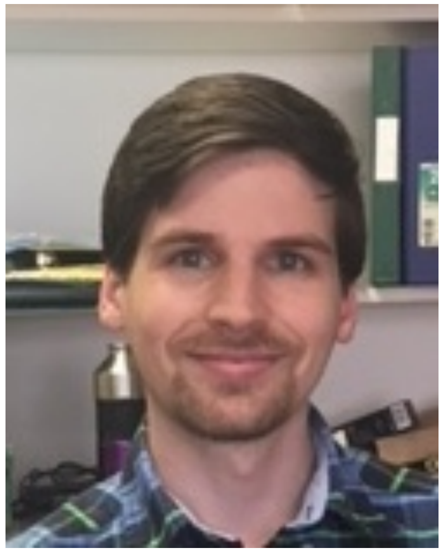
"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]

Tyler Blanton, Fernando Romero-López & SRS:



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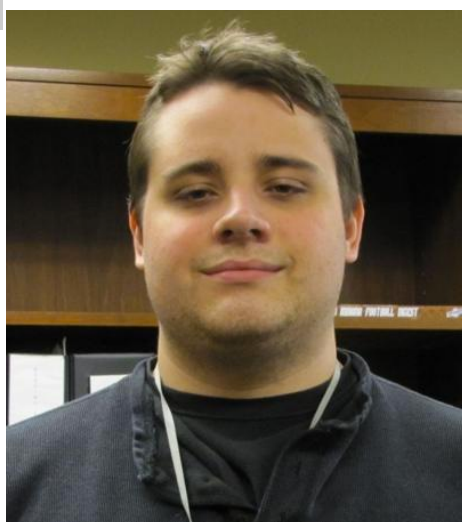
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



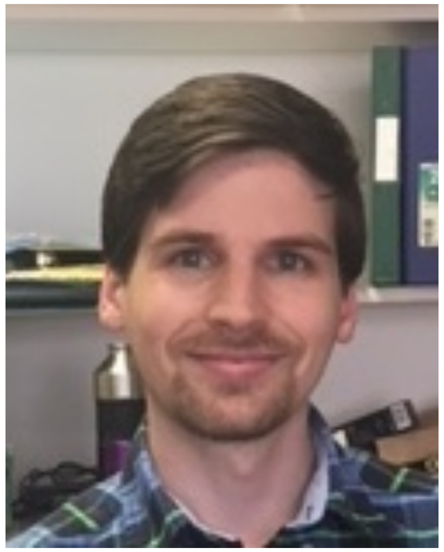
Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)



<u>Tyler Blanton & SRS:</u>

"Alternative derivation of the relativistic three-particle quantization condition," arXiv:2007.16188 (PRD) [BS20a]

"Equivalence of relativistic three-particle quantization conditions," arXiv:2007.16190 (PRD) [BS20b]

"Relativistic three-particle quantization condition for nondegenerate scalars," arXiv:2011.05520 (PRD)

"Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems," arXiv:2105.12904 (PRD)

<u>Tyler Blanton, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

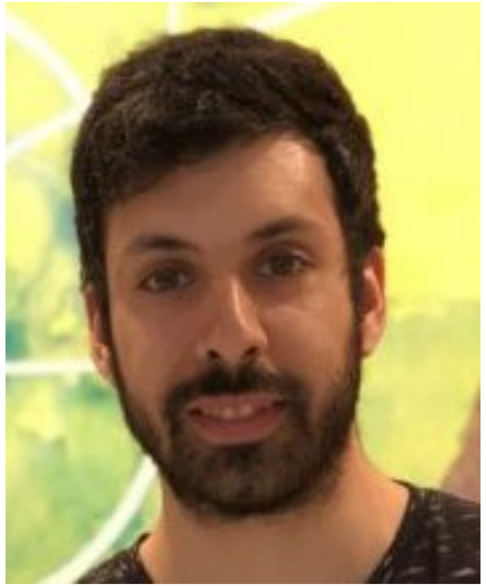
"$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD," arXiv:2106.05590 (JHEP)

<u>Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS</u>

"Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD," arXiv:2302.13587

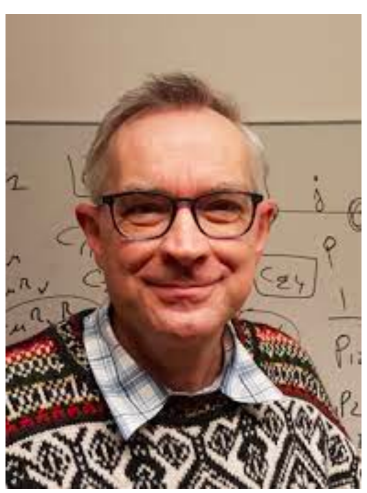


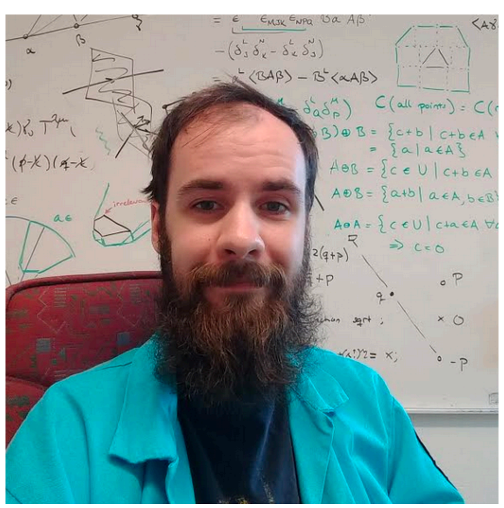




Zach Draper, Max Hansen, Fernando Romero-López & SRS:
"Three relativistic neutrons in a finite volume," arXiv:2303.10219

Jorge Baeza-Ballesteros, Johan Bijnens, Tomas Husek, Fernando Romero-López, SRS & Mattias Sjö: **"The isospin-3 three-particle K-matrix at NLO in ChPT," arXiv:2303.13206**





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- A. Jackura et al., 2010.09820 , PRD [Solving s-wave RFT integral equations in presence of bound states]
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- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

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- M. Fischer et al., 2008.03035, Eur.Phys.J.C [$2\pi^+$ & $3\pi^+$ at physical masses]
- M. Garofolo et al., 2211.05605, JHEP [3-body resonances in φ^4 theory]

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- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2211.10126, JHEP [3-particle Lellouch-Lüscher formalism in moving frames
- R. Bubna, F. Müller, A. Rusetsky, 2304.13635 [Finite-volume energy shift of the three-nucleon ground state]

Alternate 3-particle approaches

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- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
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- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
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- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
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- D. Dasadivan et al., 2112.03355, PRD [pole position of $a_1(1260)$ in a unitary framework]

★ **HALQCD approach**

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