From the femtouniverse toward the real world, via anomalies and twists

Erich Poppitz U of Toronto

various works over the years ('18-'23) with

Mohamed Anber (Durham)

Andrew Cox, F. David Wandler (Toronto)

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$= R^{1,3}$

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- **importance of symmetries in QFT**

- **'t Hooft anomalies since1980's: e.g. preons, Seiberg dualities**

- **new developments !**

THIS TALK

higher-form/discrete: *Gaiotto, Kapustin, Komargodski, Seiberg 2014-…*

"GKKS+"

TWO PARTS

MAIN: **mixed 0-form/1-form anomaly "new" vs "old"**

IN THE FORM OF COMMENTS: **anomalies, nonpertubative semiclassical dynamics, and the large volume** *("real world")* **limit**

1-FORM CENTER SYMMETRY AND 0-FORM DISCRETE SYMMETRY

1. pure 4d YM, any G, $\theta = 0 \text{ or } \pi$

parity at $\theta = 0, \pi$

2. 4d N=1 SYM, any G, *or* ***G with*** n_f ***massless adjoint Weyl****, or…*

discrete chiral $Z_{2Nn_f}^{(0)}$ for $G = SU(N) : U(1) \to Z_{2Nn_f}$

in each case, 1-FORM CENTER, e.g. $Z_N^{(1)}$ for SU(N), act on Wilson loops

in each case, 0-FORM involves 2π shift of θ angle (in YM, only at $\theta = \pi$)

explicitly:

1-FORM CENTER SYMMETRY AND 0-FORM DISCRETE SYMMETRY

$\left(\int d^3x\, K^0 = \frac{1}{8\pi^2}\int \mathrm{tr}\left(A\wedge F - \frac{i}{3}A^3\right)\right)$

1. pure 4d YM, any G, $\theta = 0$ or π

parity at $\theta = 0,\, \pi$

$$\hat{P}_\pi = \hat{V}_{2\pi}\hat{P}_0 \qquad \hat{V}_{2\pi} = e^{i\, 2\pi \int d^3x \hat{K}^0}$$

2. 4d N=1 SYM, any G, *or* ***G with*** n_f ***massless adjoint Weyl***, *or…*

discrete chiral $Z^{(0)}_{2Nn_f}$

conserved non-gauge invariant U(1) charge

$$\hat{Q}_5 = \int d^3x \hat{J}^0_5 = \int d^3x \hat{j}^0_f - 2n_f N \int d^3x \hat{K}^0$$

$$\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{i\frac{2\pi}{2n_f N}\hat{Q}_5} = e^{i\frac{2\pi}{2n_f N}\int d^3x \hat{j}^0_f}\ \hat{V}^{-1}_{2\pi}$$

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1-FORM CENTER SYMMETRY AND 0-FORM DISCRETE SYMMETRY

GKKS+ have shown that these symmetries have a mixed anomaly

usually considered in the Euclidean path integral:

gauge center -> find that discrete symmetry disappears

THIS TALK: MIXED ANOMALY IN HILBERT SPACE ON TORUS

- desire to understand a new observation from different view points

Hamiltonian was particularly useful in 2d, work w/ Anber 1807, 1811

- lattice is usually a torus

- this study reveals connections to "old" work, somewhat unknown/unappreciated!

- last but not least: allows for simple understanding of anomaly, e.g. w/out "$\mathscr{P}(B^{(2)})$"

MAIN RESULT

anomaly has immediate consequences for spectrum of $\hat{H}$: exact degeneracies of "electric flux" states in appropriately twisted Hilbert space at any size torus

N.B.: different from 'topological order' (e.g. Z_2 in superconductors), where torus degeneracy only in "topological scaling limit," neglect tunnelling at $V < \infty$

outline

1. **mixed 0-form/1-form anomaly in torus (T^3) Hilbert space**

   old: *1980's+…* T^3 *"femtouniverse" w/ twists*

   new: *anomaly interpretation*

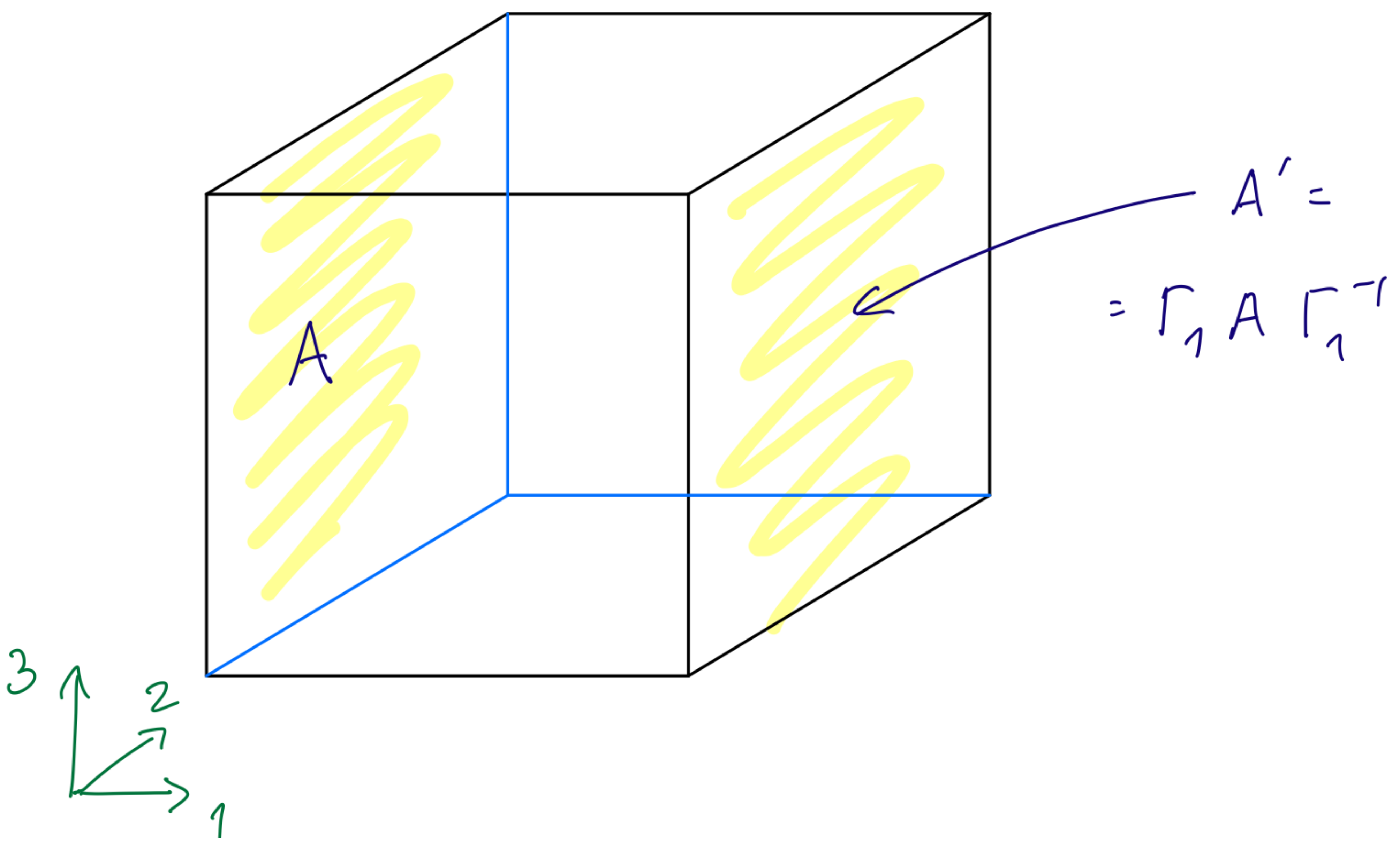
2. **consequence for spectrum of $\hat{H}$: exact degeneracies of "electric flux" states at any size T^3**

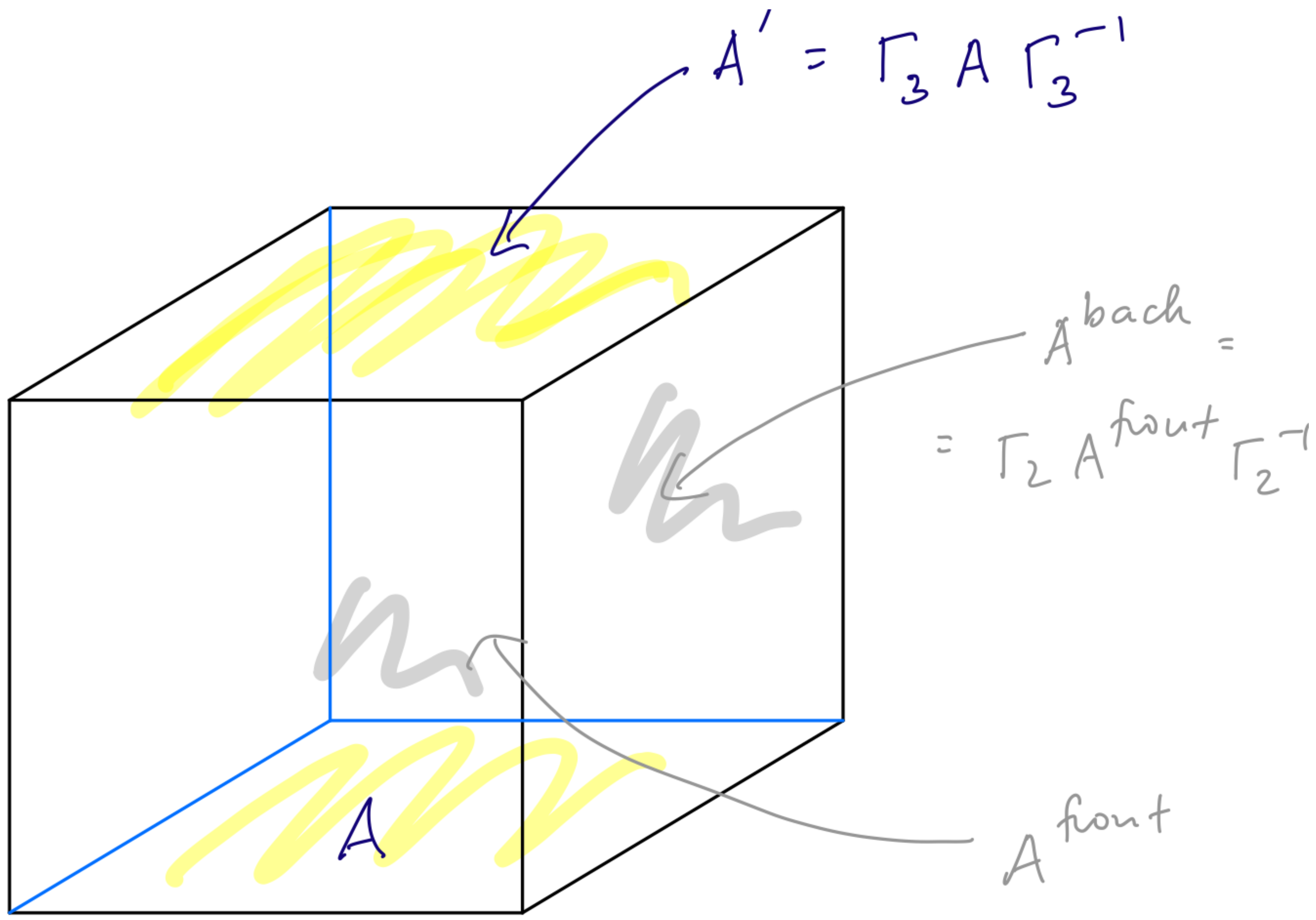
3. **implications for infinite volume phases, semiclassics, and connection to Euclidean discussions**

will skip derivations - see Cox, Wandler, EP, 2106.11442 *(nicely explained!)* -

and focus on discussing the implications (mostly on example of SU(N) at $\theta = \pi$)

framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying ***'t Hooft twisted boundary conditions***





$$\Gamma_i \Gamma_j = e^{i\frac{2\pi}{N} n_{ij}} \Gamma_j \Gamma_i$$

$$= e^{i\frac{2\pi}{N} \epsilon_{ijk} m_k} \Gamma_j \Gamma_i$$

constant-twist (Γ_i) - gauge ***(the good one!)***

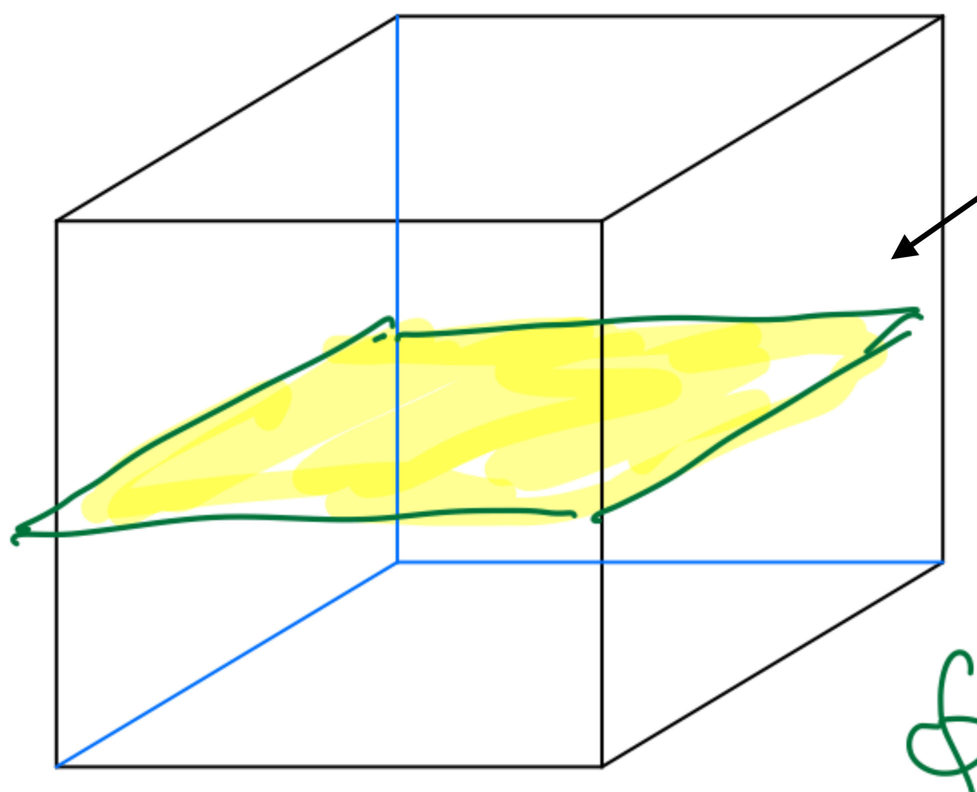
the gauge invariant data:

$$m_k \in Z \,(\mathrm{mod}\, N),\; k = 1,2,3$$

"spatial 't Hooft twists $SU(N)/Z_N$ bundle"

framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying ***'t Hooft twisted boundary conditions***

example: $m_3 = n_{12} = 1 \quad \Gamma_1 \Gamma_2 = e^{i\frac{2\pi}{N}} \Gamma_2 \Gamma_1, \Gamma_3 = 1$



= *turned on static topological 2-form Z_N gauge background in 1-2 plane* [Kapustin,Seiberg '14]

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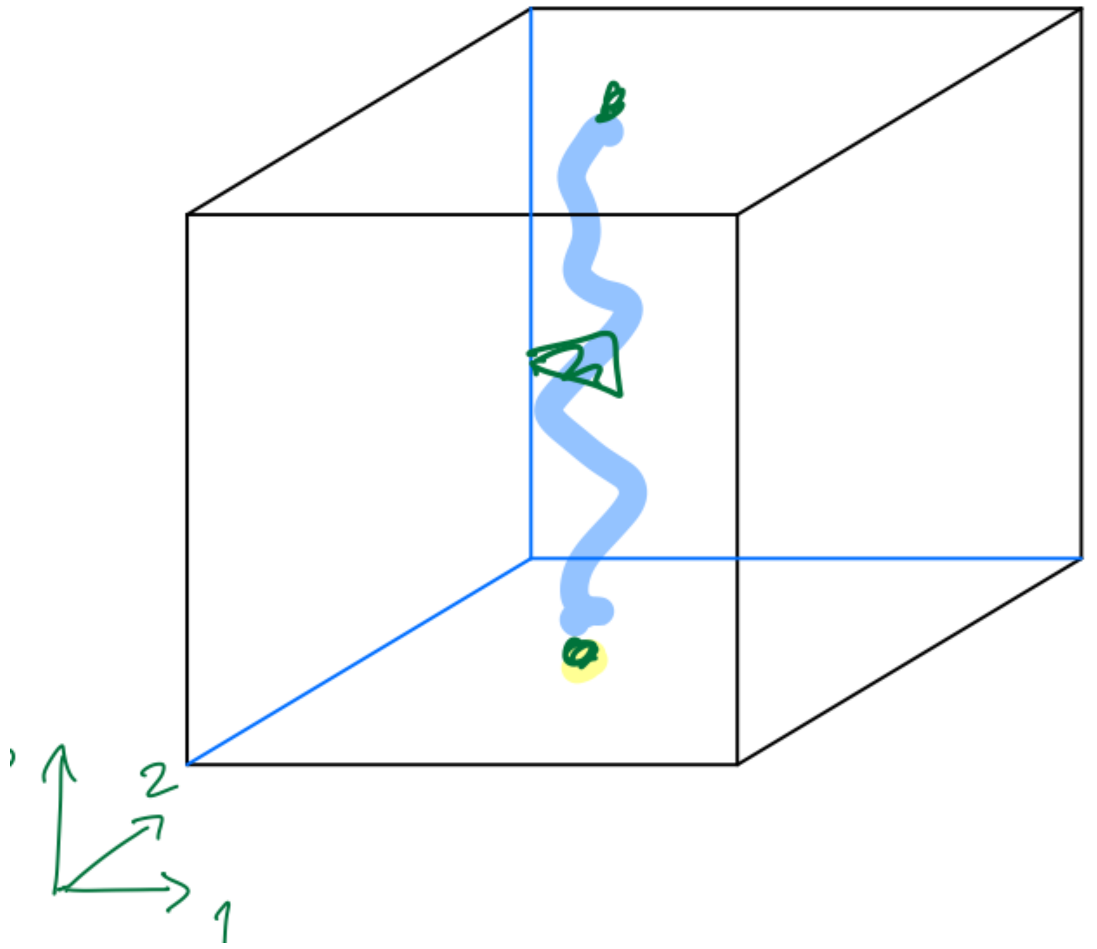

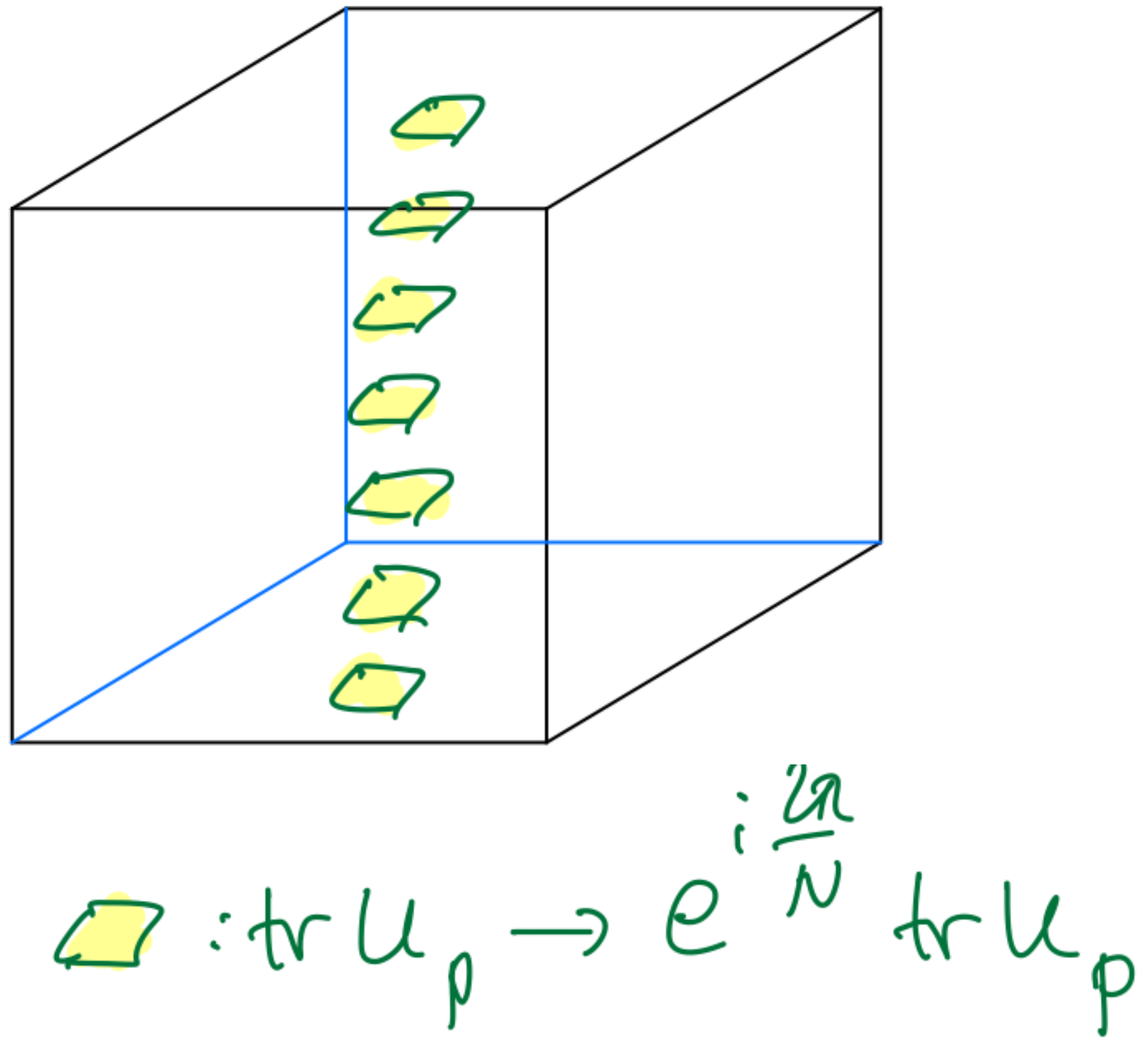

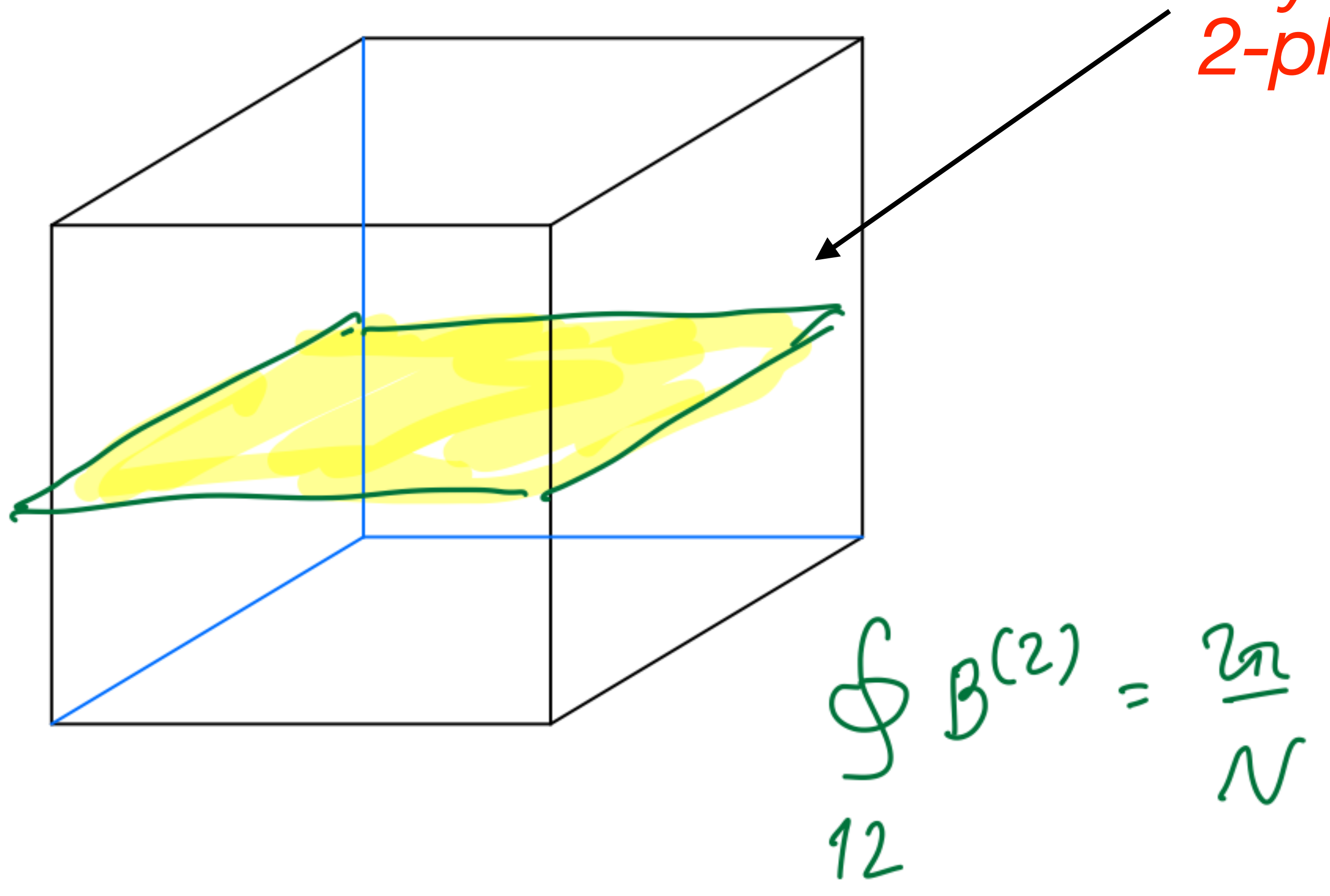

winding Dirac surface of "'t Hooft loop" = *lattice: unit center vortex in 0-3 plane* = *turned on static topological 2-form Z_N gauge background in 1-2 plane* [Kapustin,Seiberg '14]

quantizing in fixed $\vec{m}$ background
= in spatial 2-form flux gauging $Z_N^{(1)}$

framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying ***'t Hooft twisted boundary conditions***

't Hooft '81; Luscher '82; van Baal '84; Gonzalez-Arroyo; Korthals Altes '80s+…

Witten '82, '00: use for $\mathrm{tr}(-1)^F$

- center-symmetry: $\hat{T}_{l,\ l=1,2,3}$ act on winding loops $\hat{T}_l \hat{W}_k \hat{T}_l^{-1} = e^{i\frac{2\pi}{N}\delta_{kl}}\ \hat{W}_k$

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- $\hat{T}_l$ commute with Hamiltonian, generate 1-form $Z_N^{(1)}$; $\hat{T}_l$ eigenvalues $e^{i\frac{2\pi}{N}e_l} \in Z_N$

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- $\hat{T}_l$ commute with Hamiltonian, generate 1-form $Z_N^{(1)}$; $\hat{T}_l$ eigenvalues $e^{i\frac{2\pi}{N}e_l} \in Z_N$
- all eigenvectors of $\hat{H}$ also labeled by Z_N "electric flux" $\vec{e}$ *(change $\vec{e}$: act w/ W. loop)*

$$\hat{T}_l \,|\, \psi_{\vec{e}} \rangle = |\, \psi_{\vec{e}} \rangle e^{\frac{2\pi i}{N} e_l}$$

boundary conditions on T^3

$\overrightarrow{m}$ (mod N) …

discrete "magnetic flux"

eigenvalues of $\hat{T}_l$, generating 1-form Z_N

$\vec{e}$ (mod N) …

discrete "electric flux"

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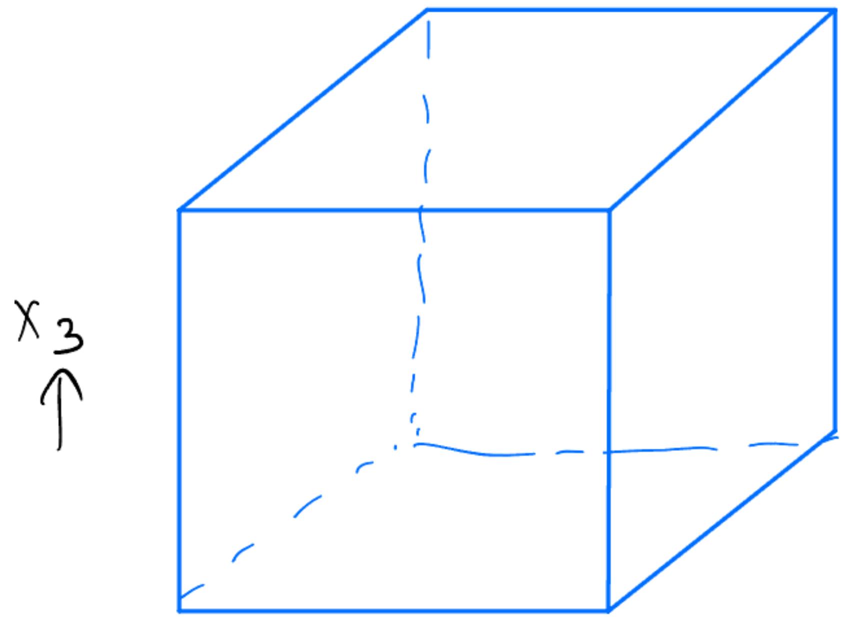
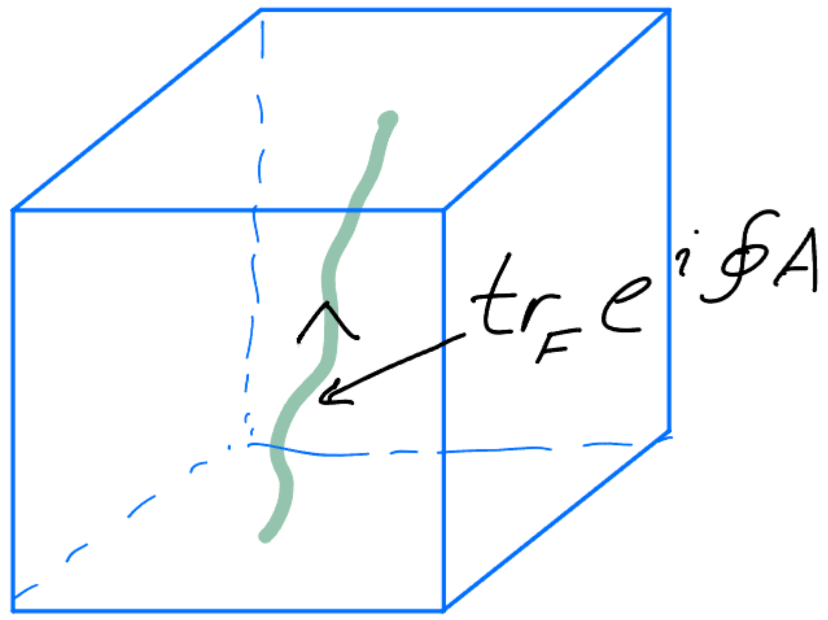
- $\hat{T}_l$ commute with Hamiltonian, generate 1-form $Z_N^{(1)}$; $\hat{T}_l$ eigenvalues $e^{i\frac{2\pi}{N}e_l} \in Z_N$

$$\hat{T}_l\,|\psi_{\vec{e}}\rangle = |\psi_{\vec{e}}\rangle e^{\frac{2\pi i}{N}e_l}$$





boundary conditions on T^3

$\overrightarrow{m}$ (mod N) …

discrete "magnetic flux"

"flux" label is due to 't Hooft does not necessarily imply nonzero gauge field strength!
(dynamical issue, twist of b.c.)

eigenvalues of $\hat{T}_l$, generating 1-form Z_N

$\vec{e}$ (mod N) …

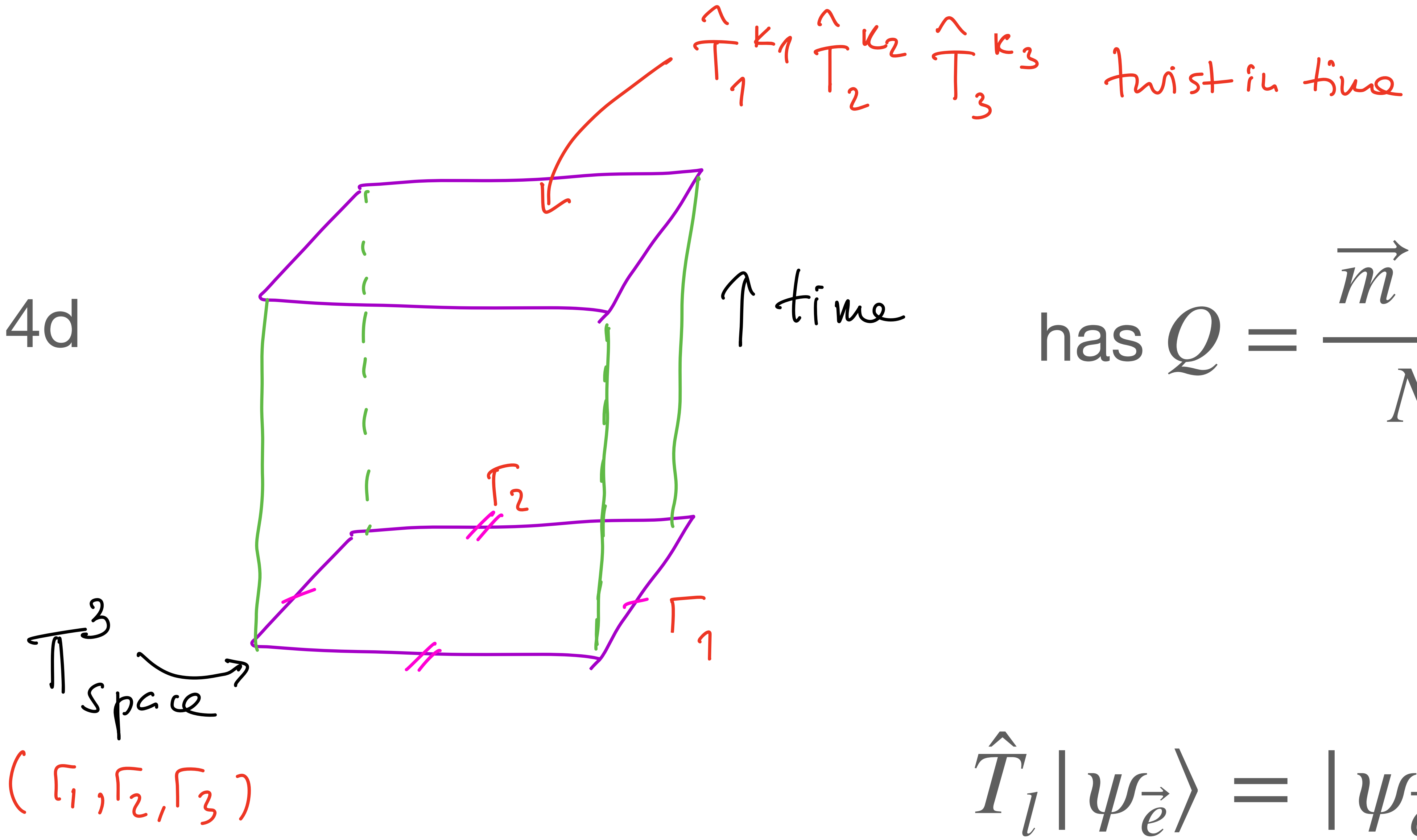
discrete "electric flux"

framework: T^3 Hilbert space: $A_0 = 0$: $\Psi[A]$ with A obeying ***'t Hooft twisted boundary conditions***

't Hooft: center-symmetry generator "along" $\vec{m}$ has fractional $T^3 \to G$ winding #

in our constant-Γ_i gauge

the gauge invariant statement is that a 4d configuration with twists as shown:



has $Q = \dfrac{\vec{m} \cdot \vec{k}}{N} (\text{mod } Z)$

$$\hat{T}_l | \psi_{\vec{e}} \rangle = | \psi_{\vec{e}} \rangle e^{\frac{2\pi i}{N}(e_l - \frac{\theta}{2\pi} m_l)}$$

argument takes too long to give in a short talk

remind you of this: ***will skip derivations*** - see Cox, Wandler, EP, 2106.11442 (nicely explained!) -

and focus on discussing the implications (mostly on example of SU(N) at $\theta = \pi$)

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't Hooft: center-symmetry generator "along" $\vec{m}$ has fractional $T^3 \to G$ winding

a picture (*J. Greensite's demand*) to illustrate fractional winding (holds in our "good" constant-Γ_i gauge)

$x \sim x+1,\ y \sim y+1,\ z \sim z+1$

$SU(2),\ \vec{m} = (0,0,1),\ \hat{T}_3(x,y,z): \mathbb{T}^3 \longrightarrow S^3 = SU(2)$

$S^3 \equiv y^\mu \in \mathbb{R}^4$

angle φ, only $\in (0,\pi)$

$S^2 \in S^3$ (const. z)

$y_1 = \cos \pi z$

$y_2 = \sin \pi z \times \sin \pi x \times 4 f(y) f(1+y)$

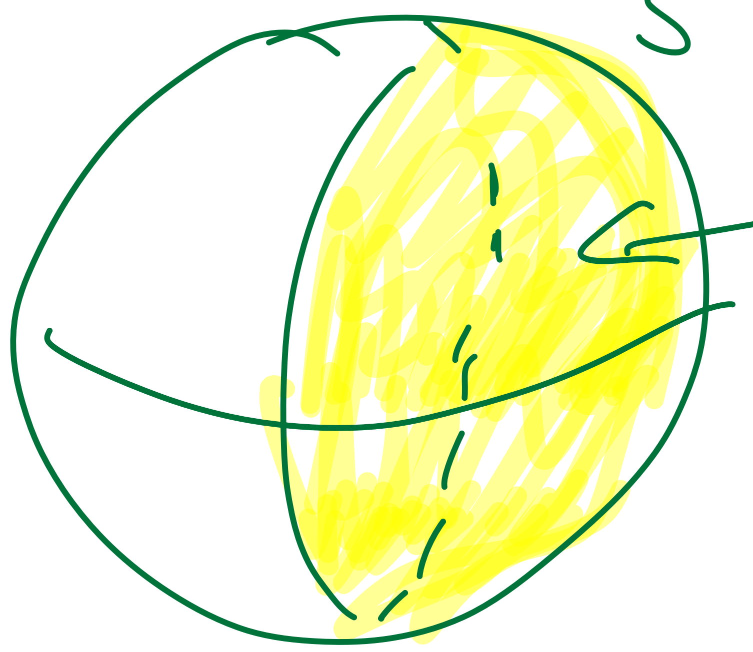
$y_3 = \sin \pi z \times \cos \pi x \times 4 f(y) f(1+y)$

$y_4 = \cos \pi z \times 2 (f^2(1+y) - f^2(y))$

$\sim \sin \theta$ (full range)

as $y \in 0, 1 \sim \cos \theta$

angle ψ, full range



$S^2 \in S^3$

image of $T^2 = (x, y)$

(explicit form of $\hat{T}_3(x,y,z)$ from Wandler, EP '22)

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immediate consequence for $\hat{V}_{2\pi} = e^{i\,2\pi \int d^3x \hat{K}^0}$

$$\hat{P}_\pi = \hat{V}_{2\pi} \hat{P}_0$$

$$\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{i \frac{2\pi}{2n_f N} \hat{Q}_5} = e^{i \frac{2\pi}{2n_f N} \int d^3x \hat{j}^0_f} \; \hat{V}^{-1}_{2\pi}$$

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immediate consequence for $\hat{V}_{2\pi} = e^{i\,2\pi \int d^3x \hat{K}^0}$

$$\hat{T}_l \; \hat{V}_{2\pi} = e^{i2\pi \frac{m_l}{N}} \; \hat{V}_{2\pi} \; \hat{T}_l$$

same as algebra in charge-N 2d Schwinger [Anber, EP '18]

SU(N) YM at $\theta = \pi$

$$\hat{T}_j \; \hat{P}_\pi = e^{\frac{2\pi i}{N} m_j} \hat{P}_\pi \; \hat{T}_j^\dagger$$

SU(N) massless QCD(adj)

$$\hat{T}_j \; \hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{-i\frac{2\pi}{N} m_j} \; \hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} \; \hat{T}_j.$$

$$\hat{P}_\pi = \hat{V}_{2\pi} \hat{P}_0$$

[Cox, Wandler, EP '21]

$$\hat{X}_{\mathbb{Z}^{(0)}_{2n_f N}} = e^{i\frac{2\pi}{2n_f N}\hat{Q}_5} = e^{i\frac{2\pi}{2n_f N}\int d^3x \hat{j}^0_f} \; \hat{V}^{-1}_{2\pi}$$

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$[\hat{T}_3, \hat{H}_{\theta=\pi}] = 0 \ , \quad [\hat{P}_\pi, \hat{H}_{\theta=\pi}] = 0 \ , \quad \hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$ ⟷ $\hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$

deformed

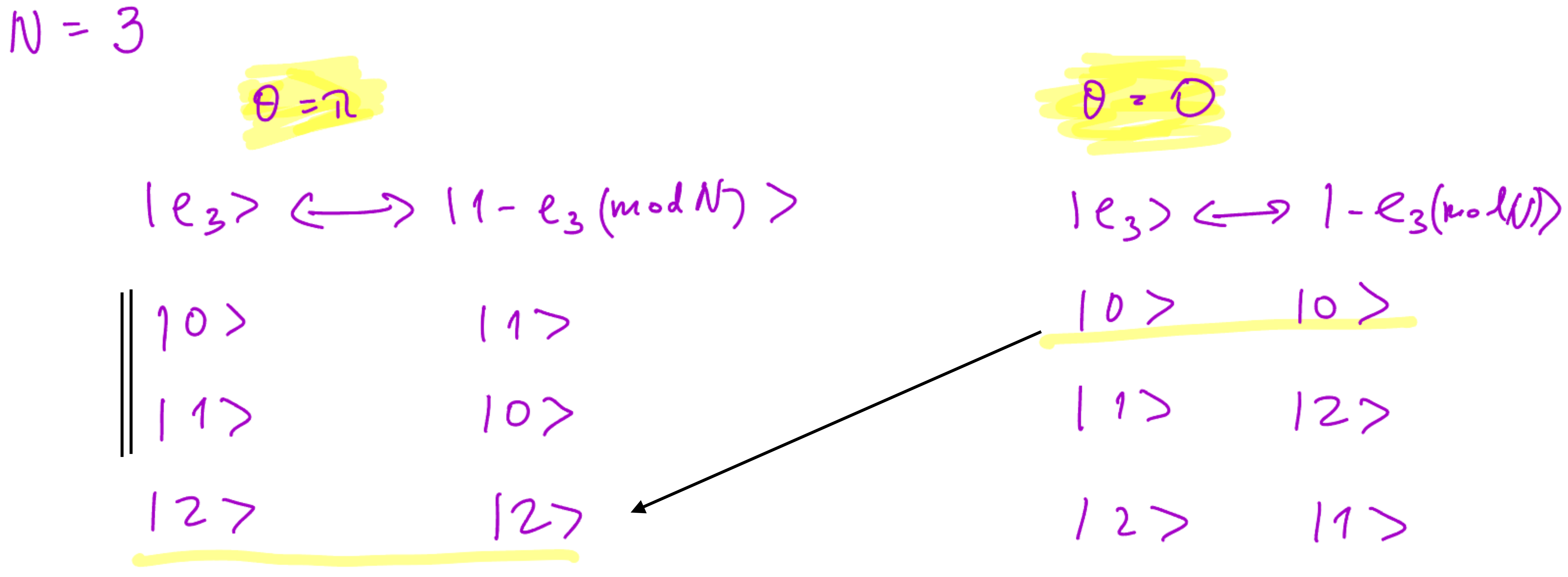
dihedral D_N (2N elements)

$\hat{P}_\pi : |E, e_3\rangle \to |E, 1 - e_3 \ (\mathrm{mod} N)\rangle$ for all E

$SU(N)$ YM, take e.g. $\vec{m} = (0,0,1)$ at $\theta = \pi$

$$\hat{P}_\pi : |E, e_3\rangle \to |E, 1 - e_3 \ (\mathrm{mod} N)\rangle$$ for all E

even N: *"anomaly"*
all states doubly degenerate!



odd N: *"global inconsistency"*

THIS IS GENERAL:

any YM at $\theta = \pi$, if center of even order, $Sp(2k+1), E_7, Spin(2k)$: double degeneracy;
if center of odd order: global inconsistency (no anomaly on T^3: $Sp(2k), Spin(2k+1)$)

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs . $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

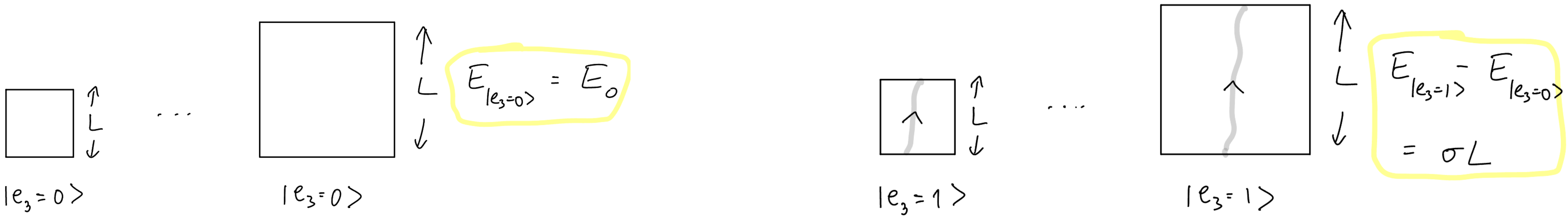
deformed *dihedral* D_{2N}

COMMENT 1:

What about real infinite-T^3 world ?

focus on **even N:** *no state mapped to itself, all states doubly degenerate!*

if **confinement?** $\theta = 0$ **expect** $e_3 = 0$ **flux to have finite E at** $L \to \infty$**, others** $E_{flux} = \sigma L$



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deformed — dihedral D_{2N}

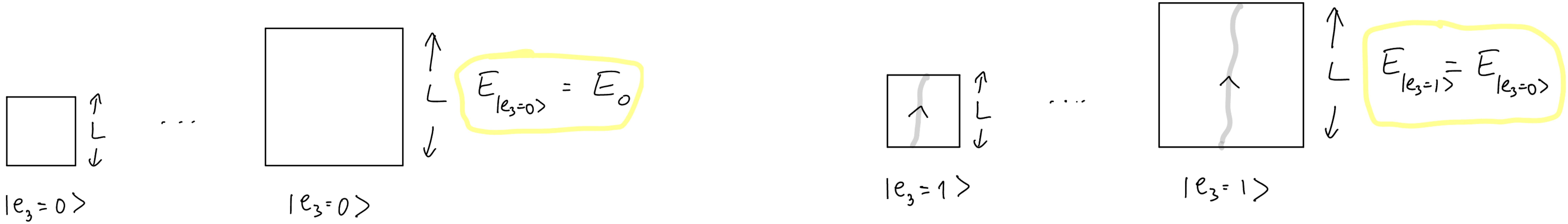
COMMENT 1:

What about real infinite-T^3 world ?

focus on **even N:** *no state mapped to itself, all states doubly degenerate!*

if **confinement?** $\theta = \pi$ **e.g.** $|e_3 = 0\rangle$ ***and*** $|e_3 = 1\rangle$ ***states of finite E at*** $L \to \infty$

clustering: ***if center preserved, parity broken:*** $|0\rangle, |1\rangle$ ***parity breaking vacua***



as in dYM [Unsal, Yaffe 2008+… coming up]*, … , lattice* [Kitano et al 2021] ?

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

deformed — dihedral D_{2N}

COMMENT 1:

What about real infinite-T^3 world ?

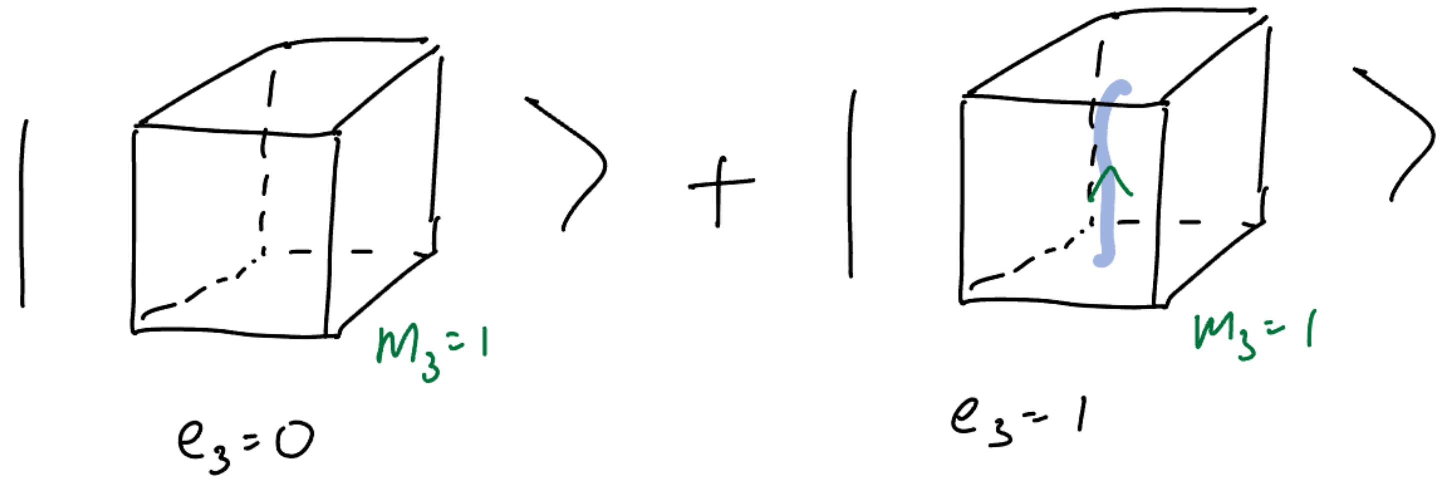
focus on **even N:** *no state mapped to itself, all states doubly degenerate!*

if **no confinement?** $\theta = \pi$ **e.g.** $|e_3 = 0\rangle$ ***and*** $|e_3 = 1\rangle$ ***states of finite E at*** $L \to \infty$

clustering: ***if parity preserved, center must be broken***

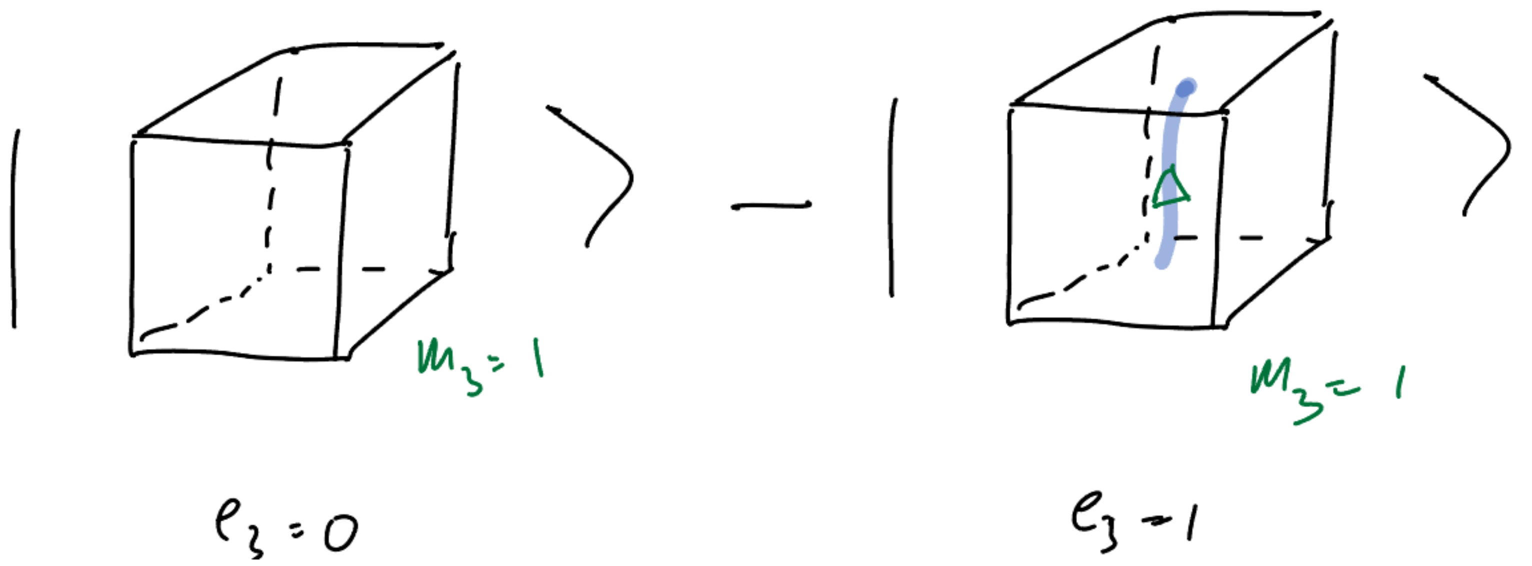
still double degeneracy at any volume

Example: 4d Georgi-Glashow $SU(2) \to U(1)$ w/ real triplet vev $v \gg 1/L$; IR-free CFT at $\theta = \pi$, at any L!



two vacua exchanged

by $\hat{T}_3$-center



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deformed *dihedral* D_{2N}

COMMENT 2:

The Euclidean connection ? connect to GKKS+

$$Z[k_3, m_3] \equiv \mathrm{tr}_{\mathscr{H}^{phys.}_{\theta=0,m_3}} (e^{-\beta \hat{H}_{\theta=\pi}} \hat{T}_3^{k_3})$$

twist by $\hat{T}_3^{k_3}$ *- path integral configurations w/* $Q_{top.} = \dfrac{k_3 m_3}{N} + n$*, summed over* $n \in Z$*.*

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs . $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

deformed dihedral D_{2N}

COMMENT 2:

The Euclidean connection ? [GKKS+]

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insert $\hat{P}_\pi \, \hat{P}_\pi = \hat{1}$ use *deformed*

$$Z[k_3, m_3] = Z[-k_3, m_3] \, e^{i\frac{2\pi}{N}k_3 m_3}$$

← “$\mathscr{P}(B^{(2)})$”

mixed anomaly in path $\int$

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

deformed — dihedral D_{2N}

COMMENT 2:

The Euclidean connection ? [GKKS+]

$$Z[k_3, m_3] \equiv \mathrm{tr}_{\mathscr{H}^{phys.}_{\theta=0,m_3}} (e^{-\beta \hat{H}_{\theta=\pi}} \hat{T}_3^{k_3})$$ insert $\hat{P}_\pi \hat{P}_\pi = \hat{1}$ use *deformed*

$$Z[k_3, m_3] = Z[-k_3, m_3] \, e^{i\frac{2\pi}{N} k_3 m_3}$$

"$\mathscr{P}(B^{(2)})$" mixed anomaly in path $\int$

solution: $Z[k_3, m_3] = e^{i\frac{\pi}{N} k_3 m_3} \, \Xi$ w/ $\Xi(k_3) = \Xi(-k_3)$

---

the two degenerate fluxes of van Baal's = TQFT, $\Xi = e^{-\beta E_{\rm vac}} \, 2\cos\frac{\pi k m_3}{N}$ (coming up)

COMMENT 3:

Discrete chiral symmetry in SYM/QCD(adj) study proceeds similarly.

Strongest constraints for SU(N): N-fold degeneracy of all electric flux states on T^3 at any L, here, anomaly: ``confinement -> chiral breaking"

[*constraints from $Z_k^{(0)}$-gravity (Cordova, Ohmori '19), assuming gap, are stronger for $G \neq SU(N)$... due to smaller rank centers*]

All my further comments below also apply for general G and SYM!

SUMMARY SO FAR:

Studied mixed 0-form/1-form anomaly: "new" vs "old"*- Hilbert space w/ twist*

main result:

Set up offers a relatively simple understanding of this type of anomaly.

Quantization in discrete $\vec{m}$ background implies exact degeneracies between $\vec{e}$-flux states, due to deformed symmetry algebra, at any finite size torus.

1. pure 4d YM, any G, $\theta = \pi$ 2. 4d N=1 SYM, any G, *or G with* n_f *adjoint Weyl, or…*

end of **MAIN**

continue **IN THE FORM OF COMMENTS about various semiclassical limits:**

femtouniverse *[small T^3, BJ… Lüscher, van Baal…'80s]* or $R^3 \times S^1$ small S^1+… *[Ünsal,…'10's]*

a word about motivation for these studies *(not that we live at $\theta = \pi$)*:

any theory of confinement should provide dynamical explanation of $\theta = \pi$ degeneracy!

here:
semiclassical understanding of confinement in various (semi ∞-volume) limits of T^3

(to be sure: …truly ∞-volume still outstanding…)

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$ vs. $\theta = 0$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger \quad \longleftrightarrow \quad \hat{T}_3 \hat{P}_0 = \hat{P}_0 \hat{T}_3^\dagger$$

deformed — *dihedral* D_{2N}

COMMENT 4:

van Baal, femtouniverse $L \ll \Lambda^{-1}$, *'84, saw this behaviour, now understood from anomaly:*

fractional instantons on $T^3 \times R$
split perturbative degeneracy of lowest "electric flux" energies

↑(lowest classical states have no gauge field strength)

$$|k\rangle = \hat{T}_3^k |0\rangle, \; k = 0,\ldots,N-1$$

$$|e\rangle \sim \sum e^{i\frac{2\pi k e}{N}} |k\rangle$$

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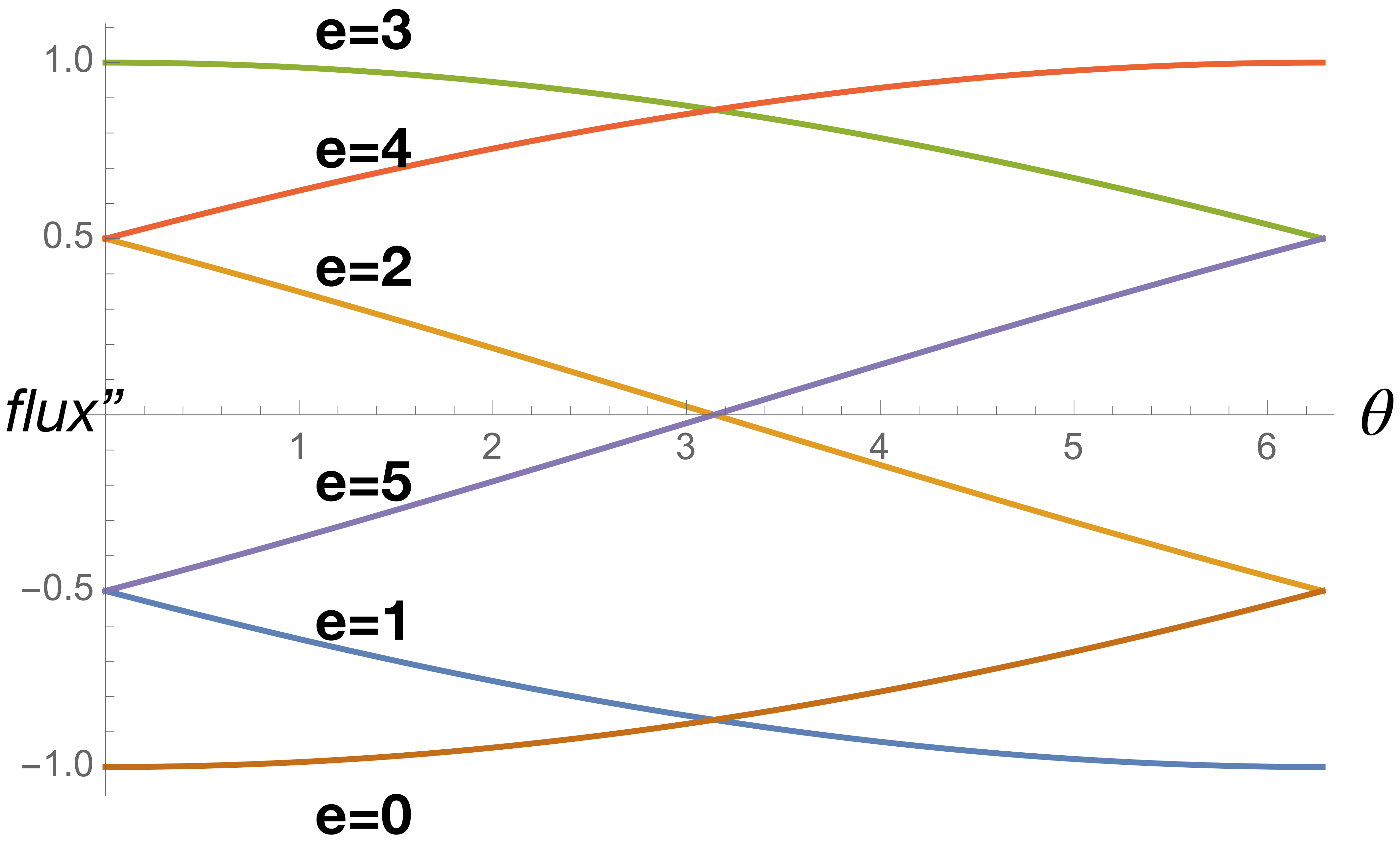
$$E(\theta, e_3) = -\frac{Ce^{-\frac{8\pi^2}{g^2 N}}}{Lg^4} \cos\left(\frac{2\pi}{N} e_3 - \frac{\theta}{N} m_3\right)$$

SU(6)



lowest "electric flux" energies

fractional instantons on $T^3 \times R$ split perturbative degeneracy of lowest "electric flux" energies

(lowest classical states have no gauge field strength)

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[van Baal, PhD thesis, '84] unpublished Ch3, picture attributed there to 't Hooft

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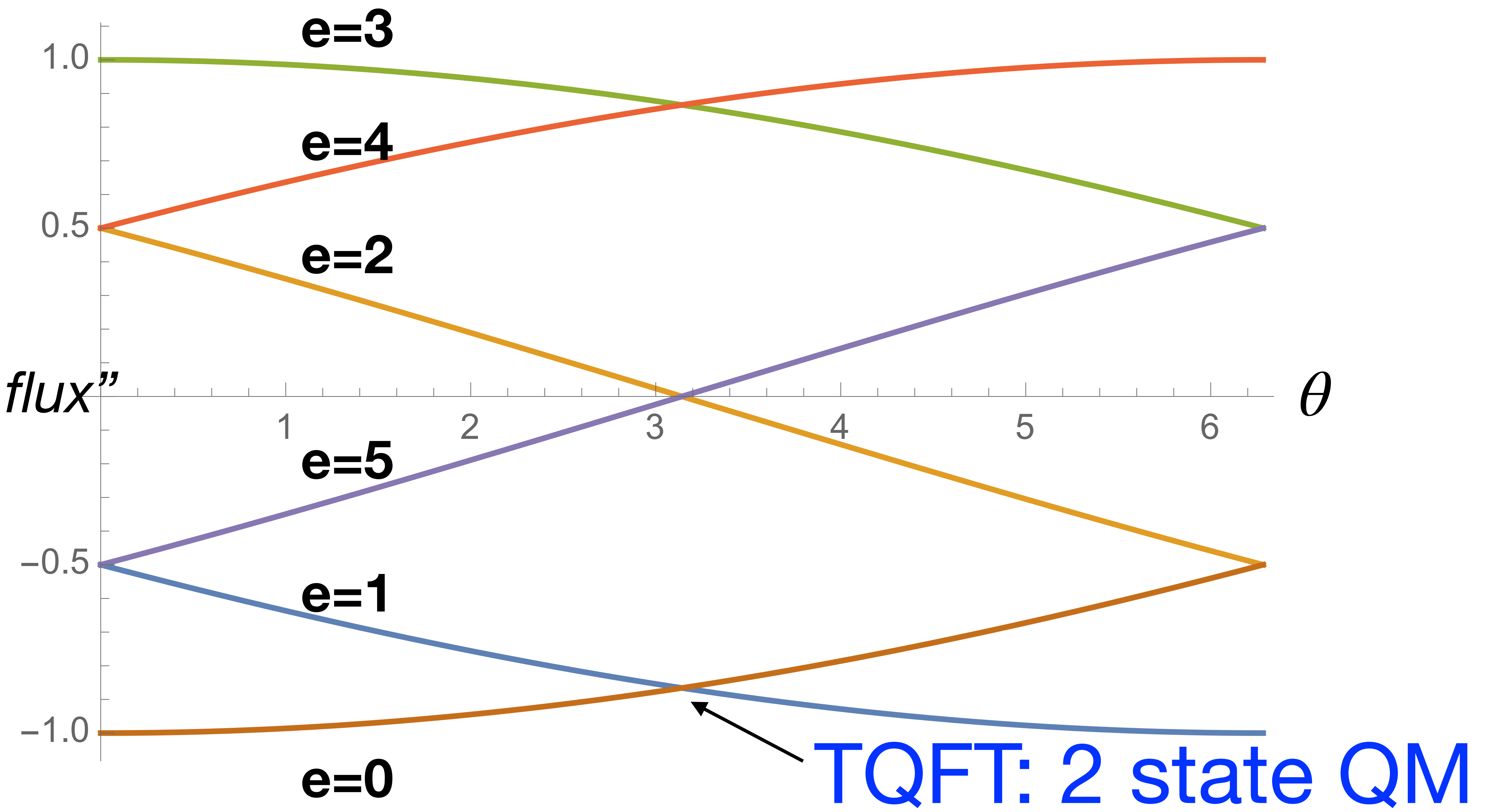
lowest "electric flux" energies



[van Baal, PhD thesis, '84] unpublished Ch3, picture attributed there to 't Hooft

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$

COMMENT 5:

A tale of two semiclassical limits

femtouniverse $L \ll \Lambda^{-1}$ ⟷ **dYM,** $R^3 \times S^1, \Lambda L N \ll 2\pi$

[van Baal, '84]

[Unsal, Yaffe '08 +]

$$E(\theta, e_3) = -\frac{Ce^{-\frac{8\pi^2}{g^2 N}}}{Lg^4} \cos\left(\frac{2\pi}{N} e_3 - \frac{\theta}{N} m_3\right)$$

$$\rho_{vac}(k,\theta) = \frac{c}{L^4} e^{-\frac{8\pi^2}{Ng^2}} \cos\left(\frac{2\pi k}{N} - \frac{\theta}{N}\right)$$

fractional instantons on $T^3 \times R$

monopole-instanton gas $R^3 \times S^1$

accident or…?

[as in Witten '79, large-N arguments, $V = \infty$ *- here, any N]*

COMMENT 5:

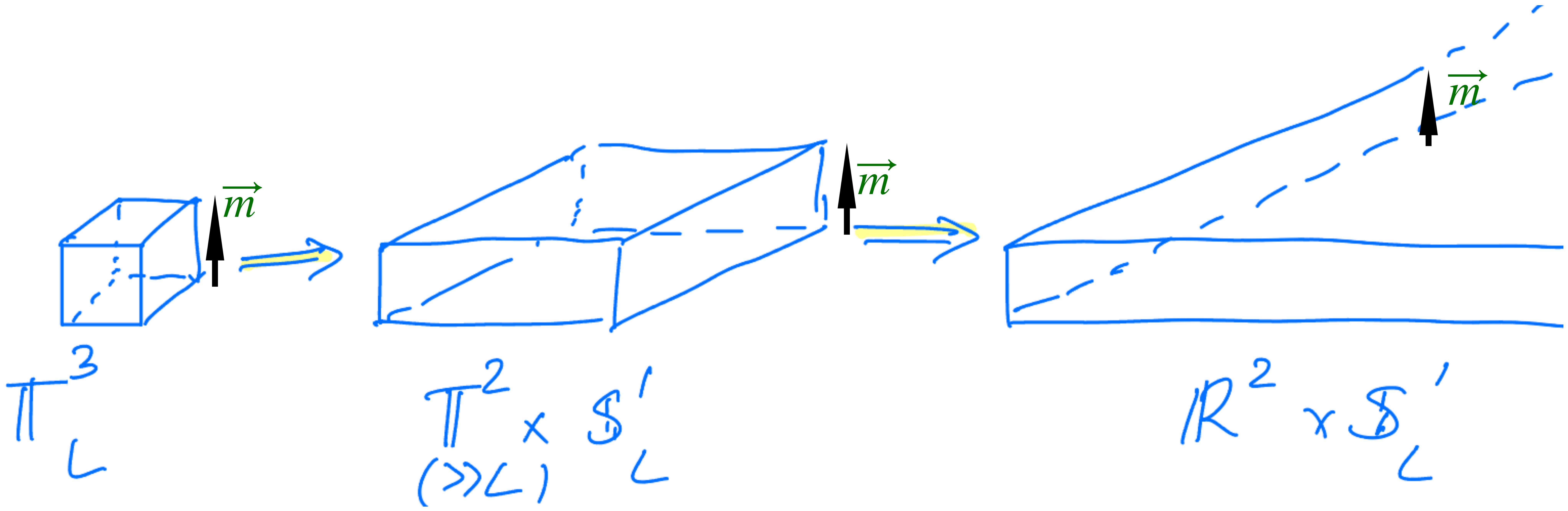
A tale of two semiclassical limits

Unsal, Yaffe '08 +

van Baal, femtouniverse $L \ll \Lambda^{-1}$ ⟷ **dYM,** $R^3 \times S^1, \Lambda L N \ll 2\pi$

dYM on $R \times T^2 \times S^1$**, with** $\vec{m}$ **through** T^2 **(along** S^1**)** [after Unsal 2020+…]

changing T^2 from $\ll \Lambda^{-1}$ to $\gg \Lambda^{-1}$, keep $L(S^1) \ll \Lambda^{-1}$



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changing T^2 from $\ll \Lambda^{-1}$ to $\gg \Lambda^{-1}$, keep $L(S^1) \ll \Lambda^{-1}$

femtouniverse

$$|k\rangle = \hat{T}_3^k |0\rangle, \, k = 0, \ldots, N-1$$

no magnetic flux $\langle k | \mathrm{tr} \hat{F}_{12} \hat{W}_3 | k \rangle = 0$

$\hat{T}_3$ "broken" classically $\langle k | \mathrm{tr} \hat{W}_3 | k \rangle \neq 0$

$\hat{T}_3$ restored by fractional instantons

then $|e\rangle \sim \sum e^{i\frac{2\pi k e}{N}} |k\rangle$ have energies $\sim e^{-\frac{8\pi^2}{g^2 N}} \cos(\frac{2\pi e}{N} - \frac{\theta}{N})$

COMMENT 5:

A tale of two semiclassical limits

Unsal, Yaffe '08 +

van Baal, femtouniverse $L \ll \Lambda^{-1}$ ⟷ **dYM,** $R^3 \times S^1$, $\Lambda L N \ll 2\pi$

dYM on $R \times T^2 \times S^1$**, with** $\vec{m}$ **through** T^2 **(along** S^1**)** [after Unsal 2020+…]

changing T^2 from $\ll \Lambda^{-1}$ to $\gg \Lambda^{-1}$, keep $L(S^1) \ll \Lambda^{-1}$

[Wandler EP '22]

femtouniverse	**dYM**
$\vert k\rangle = \hat{T}_3^k \vert 0\rangle,\ k = 0,\ldots,N-1$	$\vert k\rangle = \hat{T}_3^k \vert 0\rangle,\ k = 0,\ldots,N-1$
no magnetic flux $\langle k \vert \mathrm{tr}\hat{F}_{12}\hat{W}_3 \vert k\rangle = 0$	magnetic flux $\langle k \vert \mathrm{tr}\hat{F}_{12}\hat{W}_3 \vert k\rangle \neq 0$
$\hat{T}_3$ "broken" classically $\langle k \vert \mathrm{tr}\hat{W}_3 \vert k\rangle \neq 0$	$\hat{T}_3$ "broken" classically by flux
$\hat{T}_3$ restored by fractional instantons	$\hat{T}_3$ restored by "M" and "KK"

then $\vert e\rangle \sim \sum e^{i\frac{2\pi k e}{N}} \vert k\rangle$ have energies $\sim e^{-\frac{8\pi^2}{g^2 N}} \cos(\frac{2\pi e}{N} - \frac{\theta}{N})$

COMMENT 5:

A tale of two semiclassical limits

Unsal, Yaffe '08 +

van Baal, femtouniverse $L \ll \Lambda^{-1}$ ⟷ **dYM,** $R^3 \times S^1, \Lambda L N \ll 2\pi$

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changing T^2 from $\ll \Lambda^{-1}$ to $\gg \Lambda^{-1}$, keep $L(S^1) \ll \Lambda^{-1}$

symmetries realized identically in two limits, semiclassical objects different

[Wandler EP '22]

femtouniverse **dYM**

then $|e\rangle \sim \sum e^{i\frac{2\pi k e}{N}} |k\rangle$ have energies $\sim e^{-\frac{8\pi^2}{g^2 N}} \cos(\frac{2\pi e}{N} - \frac{\theta}{N})$

thus, these two semiclassical limits, $R \times T^3$ and $R^3 \times S^1$,
reproduce the expected vacuum structure of pure YM on R^4

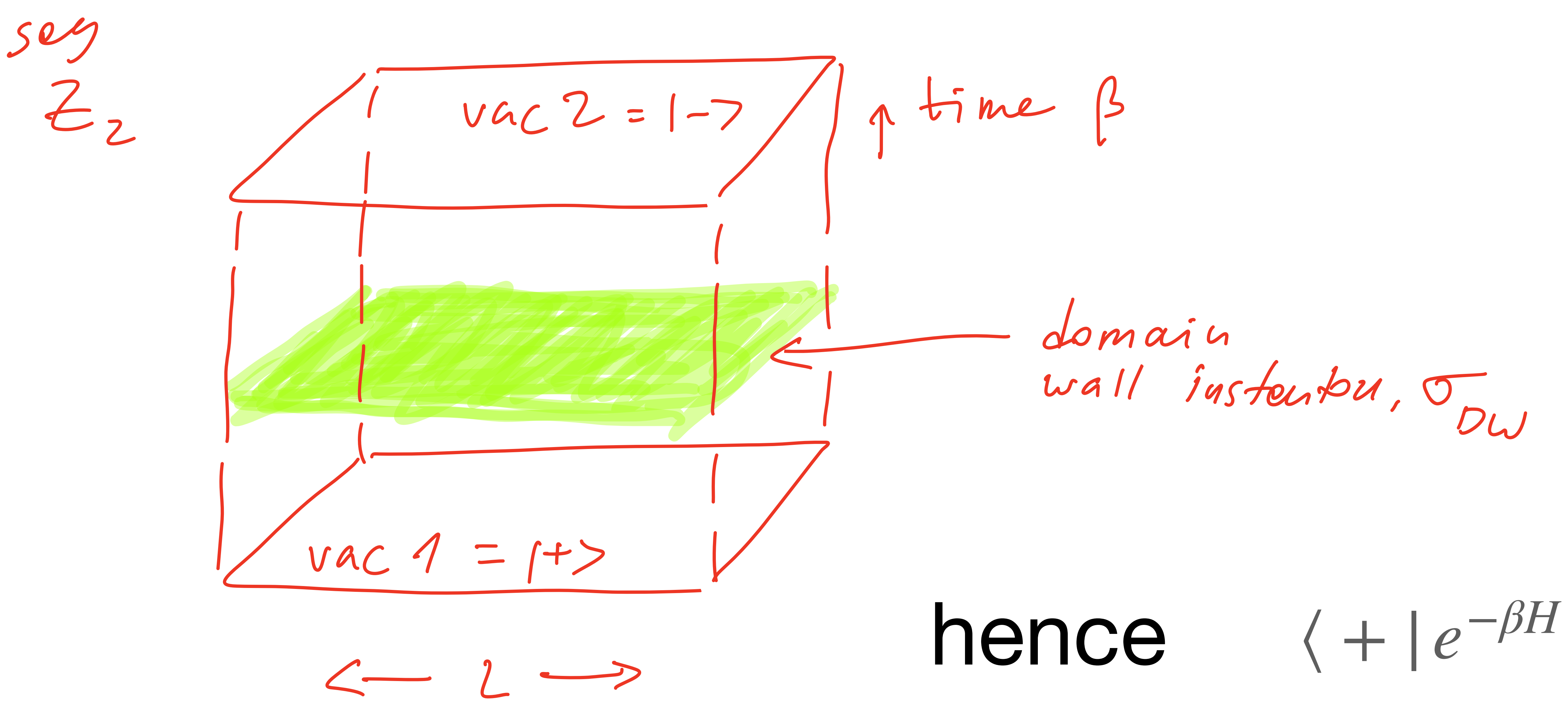
$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

COMMENT 6:

Exact degeneracy at finite volume? How come?

usually,



hence $\langle + | e^{-\beta H} | - \rangle \sim e^{-L^{d-1}\sigma_{DW}} \neq 0$ for finite *L*

we can see how this argument fails at $\theta = \pi$, semiclassically:

$SU(N)$ **YM, take e.g.** $\vec{m} = (0,0,1)$ at $\theta = \pi$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

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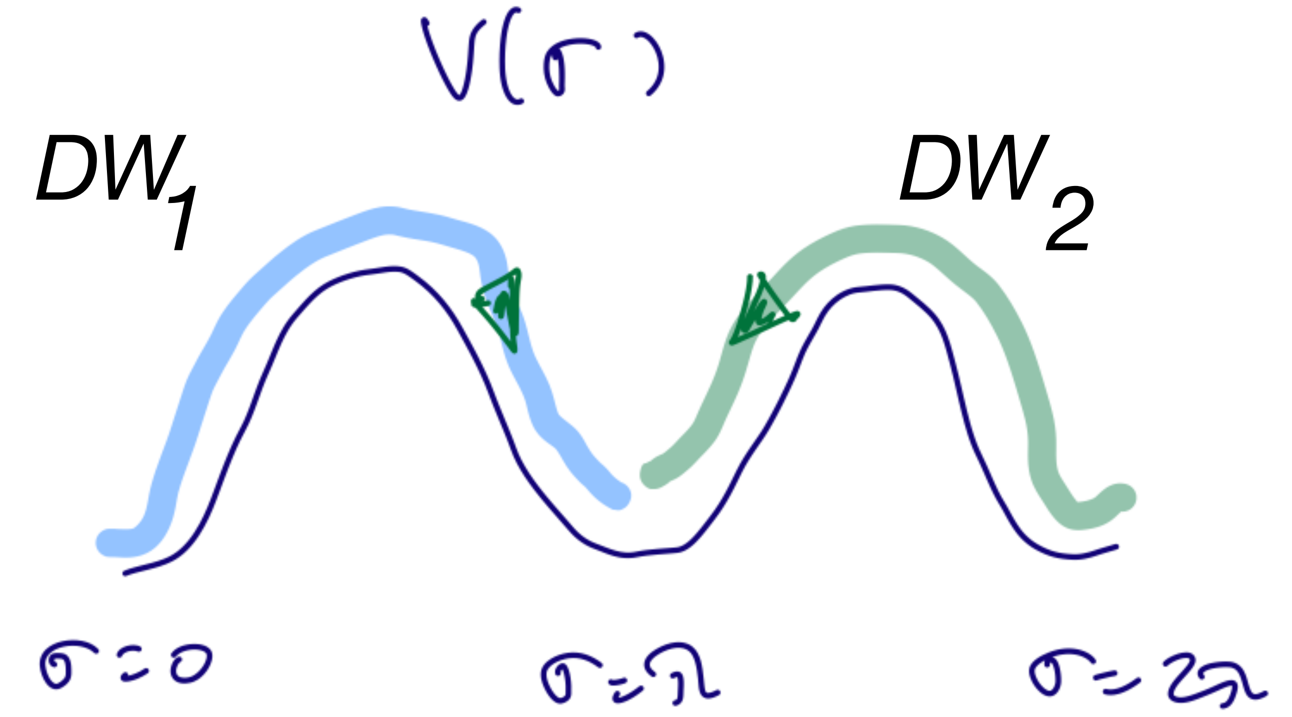
Exact degeneracy at finite volume? How come?

SU(2) dYM, $R^3 \times S^1 \; \Lambda L_{S^1} \ll 2\pi, \theta = \pi$



- two vacua (?)
- two distinct domain walls (lines!)

$SU(N)$ YM, take e.g. $\vec{m} = (0,0,1)$ at $\theta = \pi$

$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

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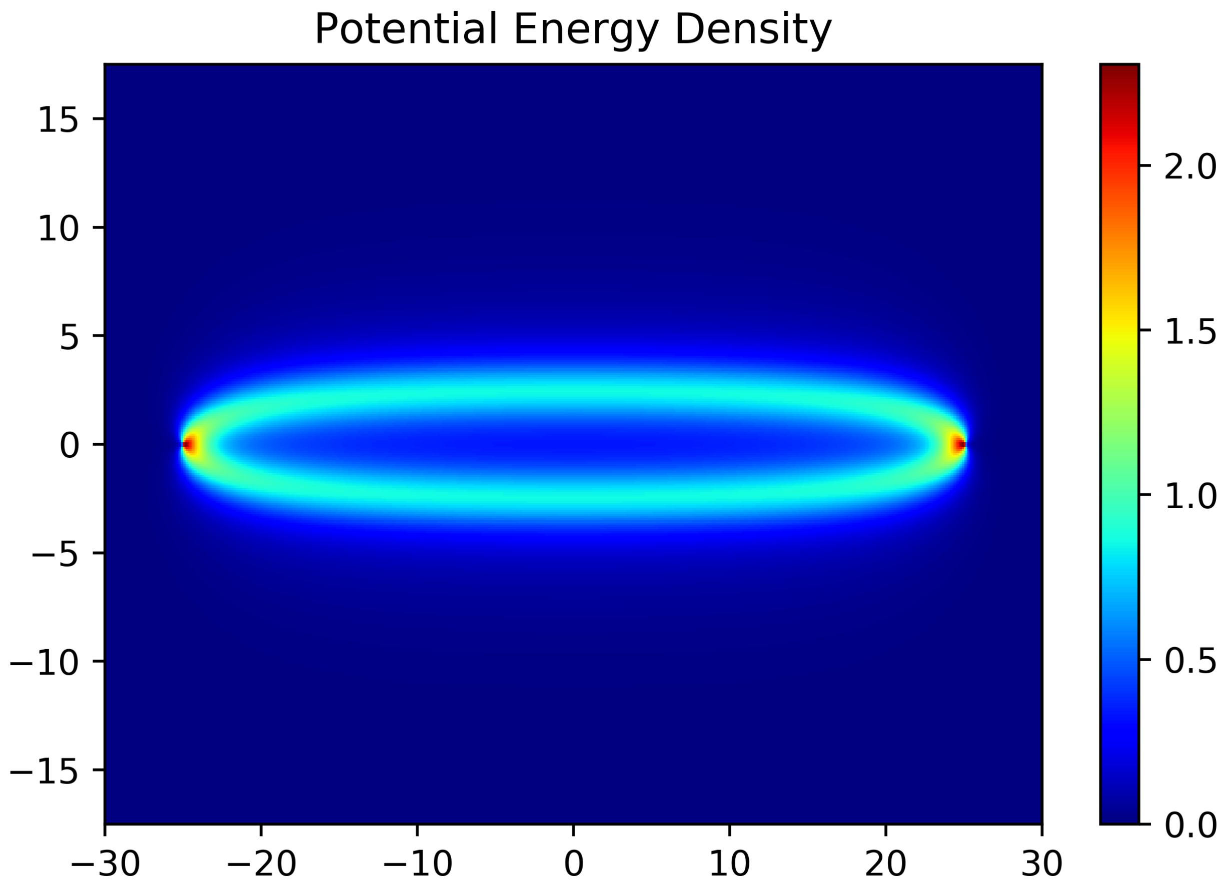
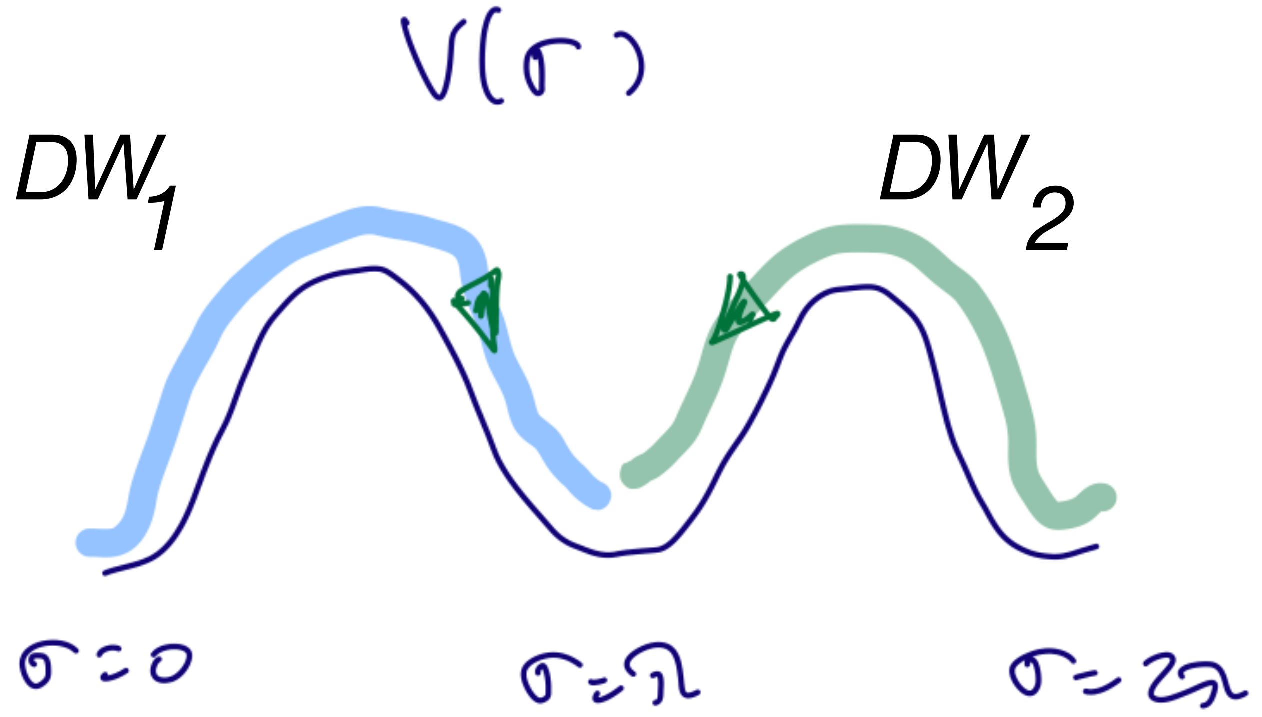
the presence of these two distinct DWs implies "double-string confinement" and "deconfinement on domain walls"… other anomaly-related phenomena found pre-anomaly!
[Anber, Sulejmanpasic, EP 2015] (will not discuss)



[from Bub, Wong, EP '19]

SU(2) dYM, $R^3 \times S^1 \; \Lambda L_{S^1} \ll 2\pi, \theta = \pi$



- two vacua ($\not{P}$)
- two distinct domain walls (lines!)

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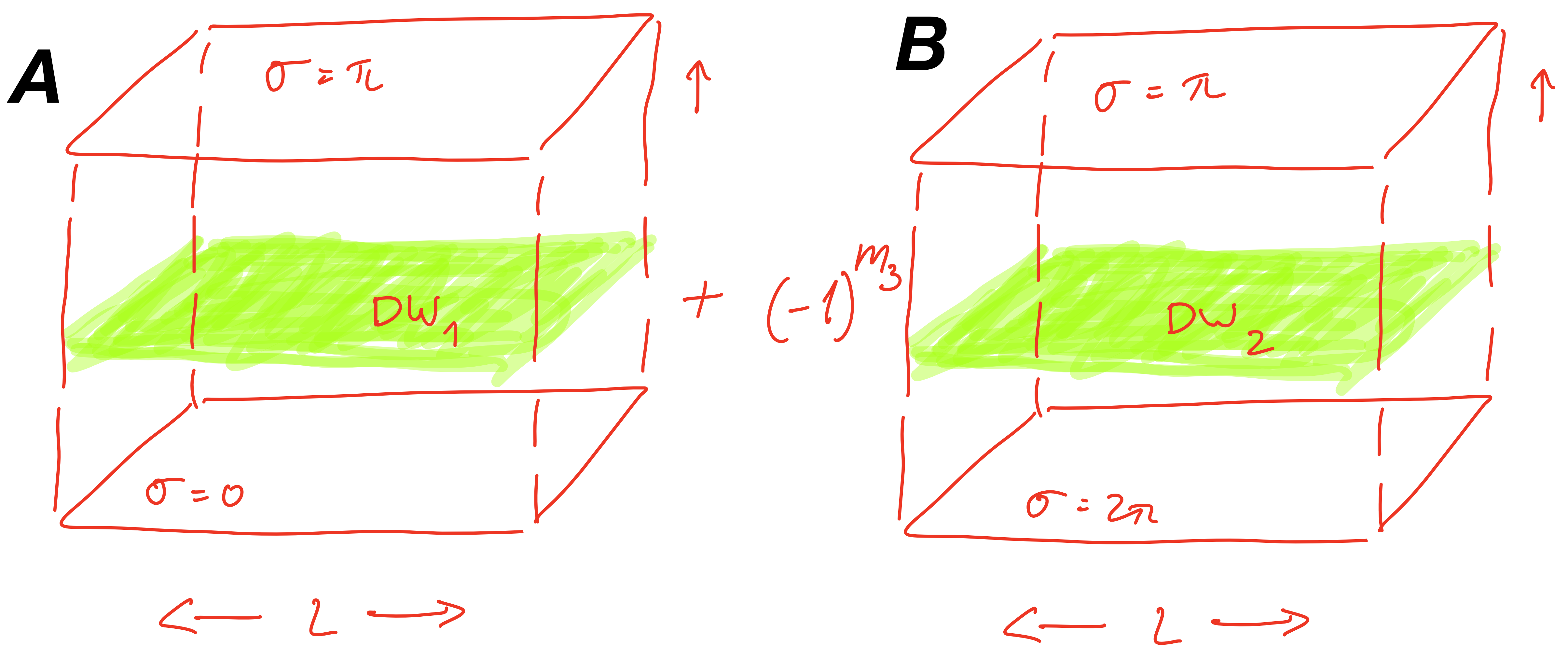
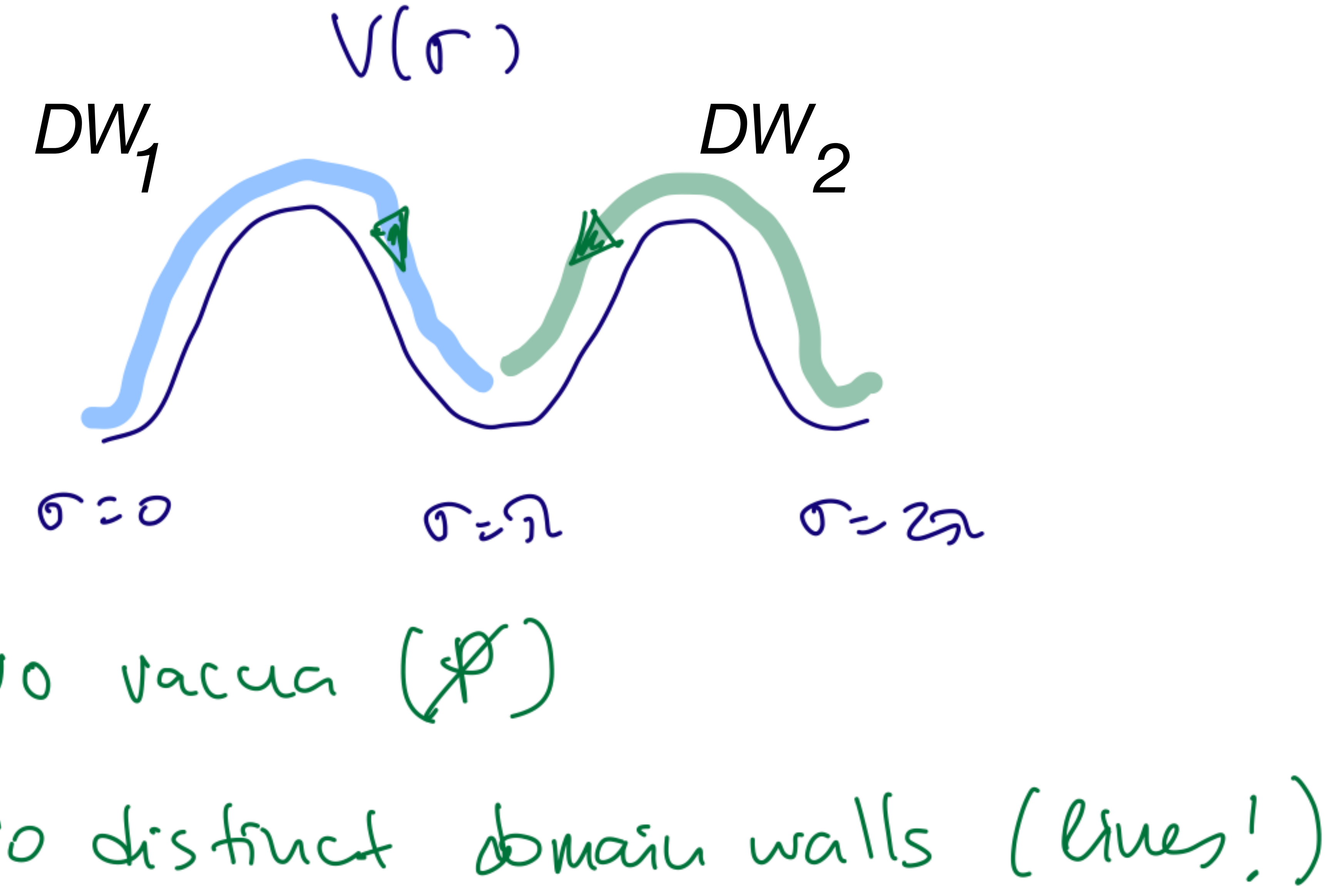
$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

COMMENT 6:

Exact degeneracy at finite volume? How come?

SU(2) **dYM:** $R \times T^2 \times S^1$, **large-**T^2, **small** S^1 **+** $m_3 = 1$





[in progress ...'23] ***-> no tunnelling even at finite volume: A+B =0 when*** $m_3 = 1$

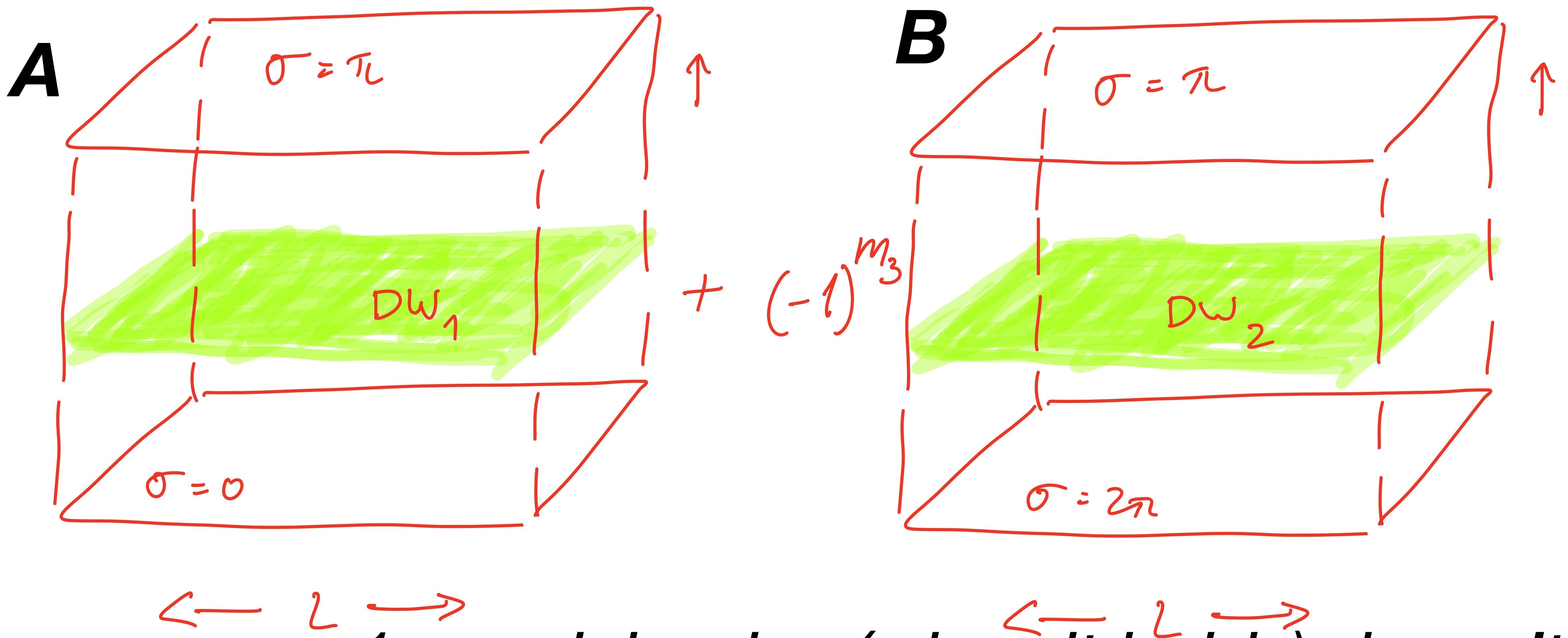
$SU(N)$ YM, take e.g. $\vec{m} = (0,0,1)$ at $\theta = \pi$

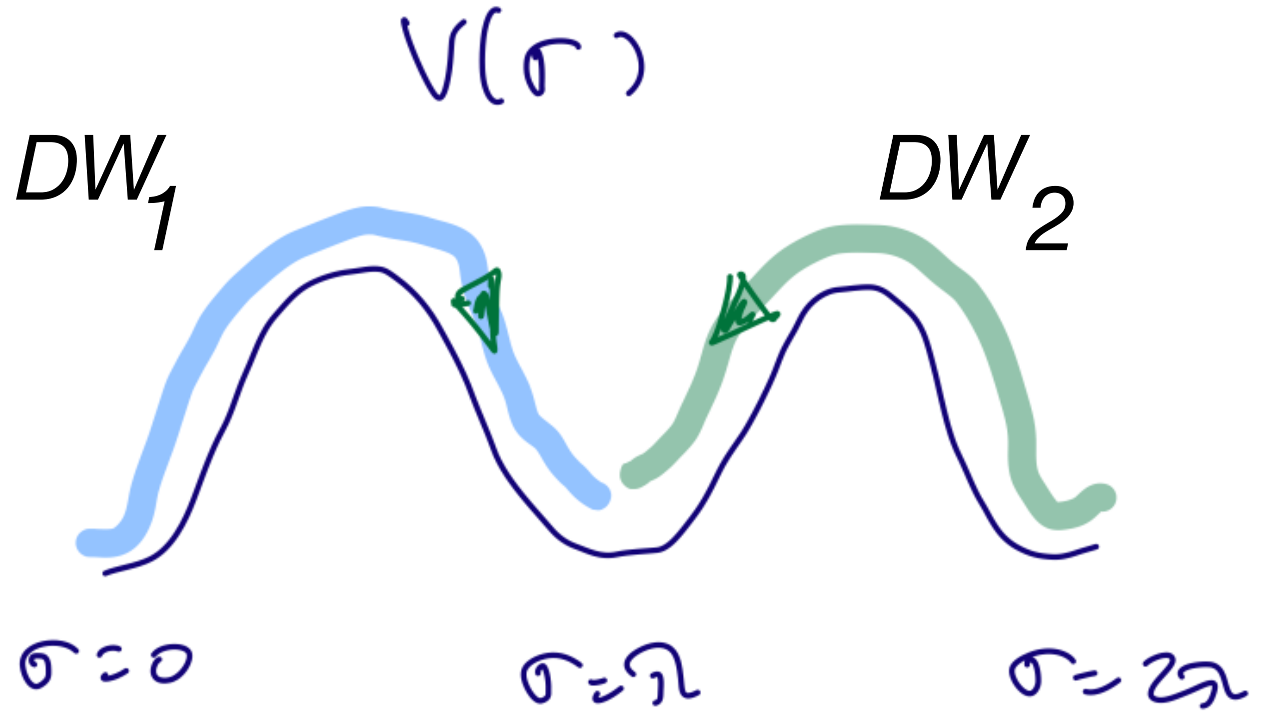
$$\hat{T}_3 \hat{P}_\pi = e^{i\frac{2\pi}{N}} \hat{P}_\pi \hat{T}_3^\dagger$$

COMMENT 6:

Exact degeneracy at finite volume? How come?

$SU(2)$ dYM: $R \times T^2 \times S^1$, large-T^2, small S^1 + $m_3 = 1$





- two vacua ($\not{P}$)
- two distinct domain walls (lines!)

1. semiclassics (when it holds) doesn't lie!
2. all of this applies as well for other SU(N>2)

SUMMARY:

Studied mixed 0-form/1-form anomaly: "new" vs "old" *- Hilbert space w/ twist*

Set up offers a relatively simple understanding of this type of anomaly.

Quantization in discrete $\vec{m}$ background implies exact degeneracies between $\vec{e}$-flux states, due to deformed symmetry algebra, at any finite size torus.

Different semiclassical limits (the femtouniverse and dYM with flux) have identical symmetry realization and produce a vacuum structure identical to that expected in infinite volume limits (YM/SYM).

FUTURE:

Exact degeneracies in $\overrightarrow{m}$ background *(trivial to implement!)* **may be useful for lattice?**

[YM $\theta = \pi$: Kitano, Matsudo, Yamada, Yamazaki '21]

Symmetry realizations in $\overrightarrow{m}$ backgrounds imply that *semiclassical objects responsible for mass gap (& confinement) in different regimes are related.*

In most cases, their nature and implications not well understood…

also Tanizaki, Ünsal '22, $R^2 \times T^2$ + incl. fundamentals!

Anber, EP, '22, SYM & gaugino condensate?

Other symmetries and anomalies?