Geometrical Methods for EFTs

Aneesh Manohar

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Applying geometrical methods to QFT computations:

- SMEFT (Standard Model Effective Field Theory)
- HEFT (Higgs Effective Field Theory)
- χ PT (Chiral Perturbation Theory)
- $O(n)$ model

Alonso, Jenkins, AM: PLB 754 (2016) 335, JHEP 08 (2016) 101

Helset, Jenkins, Manohar: PRD (2022) 116018, JHEP 02 (2023) 063

Assi, Helset, AM, Pagès, Shen: JHEP 11 (2023) 201

Jenkins, AM, Naterop, Pagès, JHEP 12 (2023) 165, JHEP 02 (2024) 131

AM, Pagès, Nepveu: HEP 05 (2024) 018 Assi, Helset, Pagès, Shen 2504.18537

Previous Work

Meetz: JMP 10 (1969) 589

Honerkamp, Meetz: PRD 3 (1971) 1996

Honerkamp: NPB 36 (1972) 130

Ecker, Honerkamp: NPB 35 (1971) 481

Alvarez-Gaumé, Freedman, Mukhi: Ann Phys 134 (1981) 85

Alvarez-Gaumé, Freedman, Mukhi: Comm Math Phys 80 (1981) 443

Gaillard: NPB 268 (1986) 669

Alonso, Kanshin, Saa, PRD 97 (2018) 035010

Helset, Martin, Trott, JHEP 03 (2020) 163

Finn, Karamitsos, Pilaftsis, 2006.058311

Finn, Karamitsos, Pilaftsis, EPJC 81 (2021) 572

Cheung, Helset, Parra-Martinez, PRD 106 (2022) 045016

Alonso, West, 2207.02050, Alonso, Kanshin, Saa, PRD 97 (2018) 035010

Cohen, Craig, Lu, Sutherland, PRL 130 (2023) 041603

Gattus, Pilaftsis, 2307.01126

Craig, Yu-Tse, Lee, Phys. Rev. Lett. 132 (2024) 6, 061602

Alminawi, Brivio, Davighi, J. Phys. A 57 (2024) 43, 435401

EFT

A Lagrangian with an expansion in a scale M (power counting)

$$L = L_{D \leq 4} + \frac{1}{M} L_5 + \frac{1}{M^2} L_6 + \cdots$$

- work to some maximum power in $1/M$ that is *finite*

$$\frac{1}{p^2 - M^2} = -\frac{1}{M^2} - \frac{p^2}{M^4} + \cdots$$

  - l.h.s. has a pole at $p^2 = M^2$ but the r.h.s. does not at any finite order
- A regularization and renormalizations scheme (dim reg) that respects the power counting in $1/M$
  - loop integrals depend only on the scales in the graph, i.e. external momenta and particle masses.

Field Redefinitions

Chisholm, NP 26 (1961) 469

Kamefuchi, O'Raifertaigh, Salam, NP 28 (1961) 529

Politzer, NPB 172 (1980) 349

- The *S*-matrix is unchanged under field redefinitions
  - On-shell Green's functions change
  - Need to include external-leg factors for invariance
- Field redefinitions consistent with EFT power counting
- Maintain symmetries such as Lorentz and gauge invariance
- Maintain locality

For example

$$\phi(x) = \phi'(x) + \frac{c_1}{M^2} \left(\phi'(x)\right)^3 + \frac{c_2}{M^2} \partial^2 \phi'(x) + \cdots$$

EOM vs Field Redefintions

Field Redefinition:

$$\phi \to \phi + \frac{1}{M^a}\, g(\phi)$$

$$S \to S + \frac{1}{M^a}\, g(\phi)\, \underbrace{\frac{\delta S}{\delta \phi}}_{EOM} + \frac{1}{2}\frac{1}{M^{2a}}\, g(\phi)\, g(\phi)\, \frac{\delta^2 S}{\delta\phi\delta\phi} + \cdots$$

The *classical* EOM is

$$\frac{\delta S}{\delta \phi} = 0$$

Eliminate EOM operators $h(\phi)\frac{\delta S}{\delta \phi}$ in the action by using a field redefinition with $g = -h$.

Have to keep all order terms in the field transformation. Using the classical EOM is incorrect at higher orders in $1/M$.

Transformations

Contact Transformations:

$$\phi'(x) = f(\phi(x))$$

Extended to

$$A'_\mu(x) = R(\phi(x))_\mu{}^\nu A_\nu(x)$$
$$\psi'_\alpha(x) = M(\phi(x))_\alpha{}^\beta \psi_\beta(x)$$

More general transformations:

$$\phi \to \phi + \frac{c_1}{M^2}\overline{\psi}\psi + \frac{c_2}{M^3}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}$$
$$A_\mu \to A_\mu + \frac{c_3}{M^2}\overline{\psi}\gamma_\mu\psi$$

or those involving derivatives.

Scalar Manifold $\mathcal{M}$

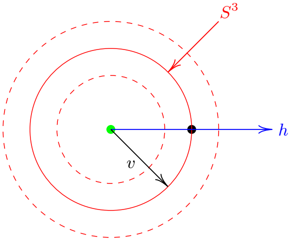
Scalar manifold $\mathcal{M}$ is the space on which scalar fields live.

The $O(n)$ model has $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_n) \in \mathbb{R}^n$.

The $O(n)$ model in the broken phase,

$$\phi_1^2 + \cdots \phi_n^2 = v^2 \qquad\qquad \boldsymbol{\phi} \in S^{n-1}$$

A point $P \in \mathcal{M}$ is specified by $(\phi_1, \cdots, \phi_n)$, so ϕ are coordinates on $\mathcal{M}$.



$O(4)$ model:

Angular directions $\varphi \in S^3$.
Radial direction h

SM with custodial symmetry

χPT

Describe pion dynamics, or more generally, Goldstone boson interactions for $G \to H$ symmetry breaking.

Scalar manifold is $\mathcal{M} = G/H$

Example: $SU(2)$ chiral perturbation theory

- Symmetry breaking $SU(2) \times SU(2) \to SU(2)$
- $\mathcal{M} = SU(2) = S^3$.
- Different parameterizations (coordinates)
  - Different Green's functions
  - Same S-matrix elements

Weinberg (1964):

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (\boldsymbol{\pi}, \sqrt{f^2 - \boldsymbol{\pi} \cdot \boldsymbol{\pi}})$$

use $\boldsymbol{\pi}$ as the independent fields.

Modern notation which generalizes to $SU(n)$ with n flavors:

$$U = u^2 \qquad u = e^{i\boldsymbol{\pi}/f} \qquad \boldsymbol{\pi} = \pi^a T^a$$

and action invariant under the $SU(n)_L \times SU(n)_R$ transformation

$$U \to R\, U\, L^\dagger$$

Standard Model

SM has spontaneous symmetry breaking $SU(2)_W \times U(1)_Y \to U(1)_{\mathrm{EM}}$.

If you assume custodial symmetry, the breaking is $SO(4) \to SO(3)_C$

$$SO(4) \sim SU(2) \times SU(2) \qquad\qquad SO(3) \sim SU(2)$$

so the breaking pattern is identical to the pion case for two flavors

[This led to the development of technicolor where $f = v = 246\ \text{GeV}$]

Three Goldstone bosons which are eaten by the $W^{\pm}, Z$, and at tree-level, $M_W = M_Z \cos\theta_W$.

Standard Model EW Symmetry Breaking

A scalar field H which transforms as $\mathbf{2}_{1/2}$ under $SU(2)_W \times U(1)_Y$.

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_2 + i\varphi_1 \\ \varphi_0 - i\varphi_3 \end{bmatrix} = \begin{bmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(v + h + i\eta) \end{bmatrix}$$

$$L = D_\mu H^\dagger D^\mu H - \lambda(H^\dagger H - v^2/2)^2$$

$\langle H \rangle = v/\sqrt{2}$ breaks the electroweak symmetry to $U(1)_{\text{EM}}$.

h is the neutral scalar with mass

$$m_h^2 = 2\lambda v^2$$

ϕ^+ and η give mass to the W and Z and are not physical states.

Standard Model

SM has two dimensionful parameters

- $v \sim 246$ GeV is the only dimensionful parameter in the SM Lagrangian at the classical level
- Λ_{QCD} due to quantum corrections. Lagrangian has $g_3(\mu)$.

The masses of the gauge bosons, leptons and quarks are proportional to v, $m_f = y_f\, v/\sqrt{2}$.

- Gauge boson masses have to be $\propto v$
- Fermion masses $\propto v$ in the SM due to theory being chiral

Hadron (proton, neutron) masses have a Λ_{QCD} piece and a quark mass piece, which is non-analytic in m_q

Higgs Sector

The Higgs boson has been found.

SM requires more than just a physical scalar particle.

$\boxed{1:}$ Yukawa couplings proportional to masses ($\sim$ 10% for some)

$\boxed{2:}$ Higgs self-couplings determined

$$V(\mathrm{h}) = \frac{1}{2} m_h^2 \mathrm{h}^2 + \frac{m_h^2}{2v} \mathrm{h}^3 + \frac{m_h^2}{8v^2} \mathrm{h}^4 = \frac{1}{2} m_h^2 \mathrm{h}^2 + 1.5 m_h \frac{\mathrm{h}^3}{3!} + 0.8 \frac{\mathrm{h}^4}{4!}$$

- $V(H)$ has two parameters, λ and v
- $v \sim 246\,\mathrm{GeV}$ determined by G_F from μ decay.
- $\lambda \sim 0.13$ determined by m_H.
- h^3 and h^4 couplings completely determined. Needs to be tested

SMEFT

A fundamental scalar field

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} i\varphi_1 + \varphi_2 \\ \varphi_0 - i\varphi_3 \end{bmatrix}$$

and an EFT using the fields of the SM.

Look for deviations from the SM generated by higher dimension operators.

Modifies both $\boxed{1}$ and $\boxed{2}$ due to dimension-six operators.

HEFT

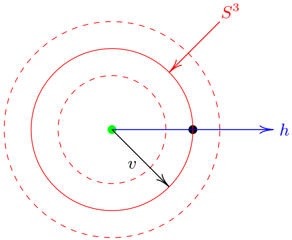
Arose from studying models where EW symmetry was broken by strong dynamics (composite Higgs models).

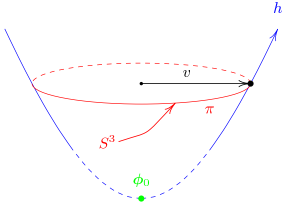
Know we have the Goldstone boson space S^3. These are eaten to give $W,\ Z$ masses.

Consider theories with symmetry breaking and a light scalar. χPT with an additional scalar field h.

SM has relations which arise from (h, φ) combining to form H, which are lost in HEFT — relations between radial and angular directions.

$\mathcal{M}$





Stack a bunch of S^3 together to contruct $\mathcal{M}$. The details depend on the specific HEFT theory, but can get some general results.

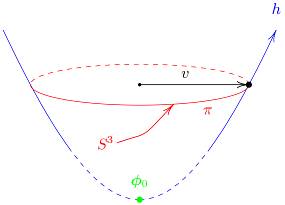
Coordinate transformations are a special case of field redefinitions

All measurable results should be coordinate invariant, i.e. geometrical.

HEFT

Alonso, Jenkins, AM, PLB 754 (2016) 355, JHEP 08 (2016) 101

$\mathcal{M}$ is a four-dimensional space with a radial direction h and angular direction S^3.



$$g_{AB} = \begin{bmatrix} F(h)^2 g_{ab}(\varphi) & 0 \\ 0 & 1 \end{bmatrix}$$

$g_{ab}(\varphi)$ is the metric on S^3 of radius v.

$F(h)$ depends on the particular HEFT model. In the SM,

$$F(h) = 1 + \frac{h}{v}$$

$A = \{h, a\}$

Riemann curvature R_{ABCD} indices either radial h or angular $a, b, \cdots$

$$R_{abcd} = \mathfrak{R}_4(h)\ \widehat{R}_{abcd} \qquad\qquad R_{ahbh} = \mathfrak{R}_{2h}(h)\ \widehat{R}_{ab}$$

$\widehat{R}_{abcd}$ and $\widehat{R}_{ab}$ are the curvature tensors of S^3 with radius v.

$$\mathfrak{R}_4(h) = \left[1 - v^2 (F'(h))^2\right] F(h)^2, \qquad \mathfrak{R}_{2h}(h) = -\frac{v^2 F(h) F''(h)}{2}$$

These vanish if $F = 1 + h/v$.

Experiments are at $h = 0$.

Goldstone boson equivalence theorem $W_L \sim \varphi$

Alonso, Jenkins, AM, PLB 754 (2016) 355, JHEP 08 (2016) 101

Terms which grow with energy are:

$$\mathcal{A}\left(W_L W_L \to W_L W_L\right) = \frac{s+t}{v^2}\,\mathfrak{R}_4(0)\,,$$
$$\mathcal{A}\left(W_L W_L \to hh\right) = -\frac{2s}{v^2}\,\mathfrak{R}_{2h}(0)\,.$$

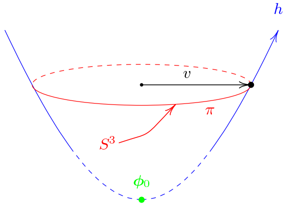
Scale of new physics is

$$M_{\text{new}} \sim \frac{4\pi v}{\sqrt{\mathfrak{R}}}$$

SM is flat, so $\mathfrak{R} = 0$, and no bad high-energy behavior.

Experiments measure the local curvature of $\mathcal{M}$ at the vacuum.

Sign fixed since $\mathfrak{R} \geq 0$ for a compact G/H space.



HEFT reduces to SMEFT if there is an $O(4)$ invariant fixed point on $\mathcal{M}$.

Alonso, Jenkins, AM, JHEP 08 (2016) 101

$$\text{SM} \subset \text{SMEFT} \subset \text{HEFT}$$

Scale of new physics and validity of EFT expansion:

Cohen, Craig, Lu, Sutherland, JHEP 03 (2021) 237, JHEP 12 (2021) 003

Geometrical methods for Scattering

Scalar Lagrangian

Helset, Jenkins, AM, 2210:08000, 2212:03253

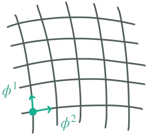


fig from Julie Pagès

Lagrangian for $\phi \in \mathcal{M}$

$$L = \frac{1}{2} g_{ab}(\phi)\, \partial_\mu \phi^a\, \partial^\mu \phi^b$$

[Huge literature on non-linear sigma models – Meetz, Honerkamp (1971); Alvarez-Gaumé, Freedman, Mukhi (1981); Gaillard (1986)]

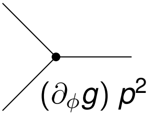
g_{ab} transforms like a metric under field redefinitions $\phi \to \phi(\phi')$ and is taken to be the metric on $\mathcal{M}$.

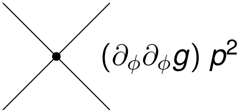
Perturbation theory about the vacuum $\phi = 0$.

$\phi\phi \to \phi\phi$

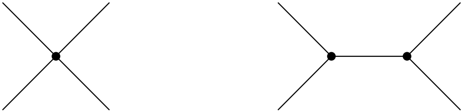
Expand g_{ab}

$$g_{ab}(\phi) = g_{ab}(0) + g_{ab,c}(0)\phi^c + \frac{1}{2} g_{ab,cd}(0)\phi^c\phi^d + \ldots$$





$\phi\phi \to \phi\phi$



$$\mathcal{A} \sim \mathcal{O}(p^2)$$

[Weinberg's power counting argument in χPT].

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad} \left(g_{db,c} + g_{dc,b} - g_{bc,d} \right)$$

$$R^a{}_{bcd} = \frac{\partial \Gamma^a_{db}}{\partial \phi^c} - \frac{\partial \Gamma^a_{cb}}{\partial \phi^d} + \Gamma^a_{ch} \Gamma^h_{db} - \Gamma^a_{dh} \Gamma^h_{cb}$$

$$R \sim \partial \Gamma + \Gamma\ \Gamma \sim (\partial_\phi \partial_\phi g) + (\partial_\phi g)^2$$

S-matrix for $\phi\phi \to \phi\phi$

Look at $\phi\phi \to \phi\phi$ and treat all indices as incoming.

$$\mathcal{A}_{abcd} = R_{abcd} s_{ac} + R_{acbd} s_{ab} \qquad\qquad s_{ij} = (p_i + p_j)^2$$

Amplitude is Bose-symmetric using the symmetry properties of R_{abcd}:

$$\begin{aligned} R_{abcd} &= -R_{bacd} \\ R_{abcd} &= R_{cdab} \\ R_{abcd} + R_{acdb} + R_{adbc} &= 0 \end{aligned}$$

and

$$p_a + p_b + p_c + p_d = 0$$

$$\implies s + t + u = 0$$

With a potential $V(\phi) \implies$ terms depend on covariant derivatives of V.

Scalars + Gauge

Helset, Jenkins, AM, 2210:08000, 2212:03253

$$L = \frac{1}{2} h_{IJ}(\phi) \partial_\mu \phi^I \, \partial_\mu \phi^J - \frac{1}{4} g_{AB}(\phi) F^A_{\mu\nu} F^{B\mu\nu} - V(\phi)$$

Combine into a single field

$$\Phi^a = \begin{bmatrix} \phi^I \\ A^A_\mu \end{bmatrix} \qquad\qquad a = \{I, (A, \mu)\}$$

and a metric

$$g_{ab} = \begin{bmatrix} h_{IJ} & 0 \\ 0 & -\eta_{\mu\nu} g_{AB} \end{bmatrix}$$

Compute as before. Can have Γ and R with indices in both scalar and gauge space.

Scalars + Gauge

The Riemann tensor satisfies the Bianchi identities as before. For an external gauge particle

$$\epsilon_A^\mu R_{(\mu,A)bcd} \qquad\qquad p_A^\mu R_{(\mu,A)bcd} = 0$$

Compute the tree-level on-shell amplitude

$$\mathcal{A}_{ijkl} = R_{ijkl}\, s_{ik} + R_{ikjl}\, s_{ij}\,,$$

Describes $\phi\phi \to \phi\phi$, $\phi A \to \phi A$, etc.

Scalars + Gauge + Fermions

Finn, Karamitsos, Pilaftsis, 2006.058311

Assi, Helset, AM, Pagès, Shen 2307.03187

Assi, Helset, Pagès, Shen 2504.18537

Radiative Corrections

Radiative Corrections

Restrict to Lagrangians with two derivatives.

Extension to higher derivative terms

Cheung, Helset, Parra-Martinez, PRD 106 (2022) 045016

Cohen, Craig, Lu, Sutherland, PRL 130 (2023) 041603

Craig, Yu-Tse, Lee, Phys. Rev. Lett. 132 (2024) 6, 061602

Alminawi, Brivio, Davighi, J. Phys. A 57 (2024) 43, 435401

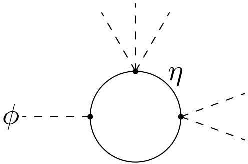
Radiative Corrections

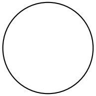
Functional method:

$$\phi \to \phi + \eta$$

and then integrate over the quantum field η.

At one-loop need order η^2 terms:





skeleton graph

't Hooft Formula

't Hooft, NPB 62 (1973) 444.

Quadratic terms in $\eta = (\eta^1, \ldots \eta^N)$ can be written as

$$L = \frac{1}{2}(\partial_\mu \eta)^T (\partial^\mu \eta) + (\partial_\mu \eta)^T N^\mu(\phi)\eta + \frac{1}{2}\eta^T X(\phi)\eta$$

with N_μ an antisymmetric matrix.

$$L = \frac{1}{2}(D_\mu \eta)^T (D^\mu \eta) + \frac{1}{2}\eta^T X \eta$$
$$D_\mu \eta = \partial_\mu \eta + N_\mu \eta$$

absorbing $N_\mu N^\mu$ into X.

This looks like an $O(N)$ gauge theory with N_μ as the gauge potential.

't Hooft Formula

$$Y_{\mu\nu} = \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu]$$

is the field-strength tensor.

One loop counterterm is

$$L_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon} \text{Tr}\left[-\frac{1}{4}X^2 - \frac{1}{24}Y_{\mu\nu}^2\right]$$

These are the only possible terms with the correct dimension allowed by $O(N)$ gauge invariance. Need to do graphs to compute numerical coefficients.

Issues

Generate non-covariant terms. The first variation of the action transforms like a vector, but the second variation does not transform like a tensor.

e.g. if $\phi(\lambda)$ is a family of field configurations, the tensors are

$$\frac{\mathrm{d}\phi^a}{\mathrm{d}\lambda} \qquad\qquad \frac{\mathrm{d}^2\phi^a}{\mathrm{d}\lambda^2} + \Gamma^a_{bc}\frac{\mathrm{d}\phi^b}{\mathrm{d}\lambda}\frac{\mathrm{d}\phi^c}{\mathrm{d}\lambda}$$

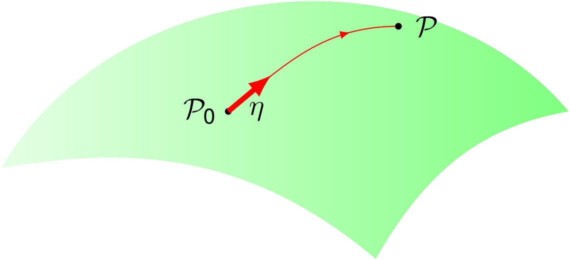
w.r.t. a variation parameter λ.

't Hooft's formula requires the kinetic term to be $(\partial_\mu\eta)^2/2$. So cannot directly be applied to EFTs.

e.g. in SMEFT, C_{HD} and $C_{H\Box}$ generate $\phi^2(\partial\eta)^2$ terms, which are not included in the 't Hooft formula.

Cannot make a field redefinition of ϕ to turn g_{ij} into δ_{ij}. The obstruction to this is the Riemann curvature tensor R_{ijkl} constructed from g_{ij}.

Riemann normal coordinates (Geodesic Coordinates)



A geodesic starting at $\mathcal{P}_0$ with velocity η that ends at $\mathcal{P}$ in unit time. Then the coordinates η of $\mathcal{P}$ are a tensor at $\mathcal{P}_0$.

$$g_{ij}(\mathcal{P}_0) = \delta_{ij} \qquad \Gamma^i_{jk}(\mathcal{P}_0) = 0 \qquad g_{ij} = \delta_{ij} - \frac{1}{3} R_{ikjl} \eta^k \eta^l + \dots$$

Quadratic in η terms in Riemann Normal Coordinates

You can compute expansion of action using covariant derivatives in a coordinate-free way.

Alonso, Jenkins, AM JHEP 08 (2016) 101

The quadratic in η terms are:

$$L = \frac{1}{2} g_{ij} \left(\mathscr{D}_\mu \eta\right)^i \left(\mathscr{D}^\mu \eta\right)^j - \frac{1}{2} R_{ikjl} \left(D_\mu \phi\right)^k \left(D^\mu \phi\right)^l \eta^i \eta^j - \frac{1}{2} \left(\nabla_i \nabla_j V\right) \eta^i \eta^j$$

with

$$\left(\mathscr{D}_\mu \eta\right)^i = \left(\partial_\mu \eta^i + \Gamma^i_{kj} \partial_\mu \phi^k \eta^j\right) + A^\beta_\mu \left(t^i_{\beta\,,j} + \Gamma^i_{jk}\, t^k_\beta\right) \eta^j$$

- $(\mathscr{D}_\mu \eta)$ is gauge and coordinate covariant, and transforms as a coordinate vector.
- Covariant derivatives of the potential, not ordinary derivatives
- Still have a non-trivial kinetic term

Everything is a tensor, so we can go to a local inertial frame (Cartan frame, i.e. use vielbeins).

$$
\begin{aligned}
g_{ij}(\phi) &= e_i^a(\phi)\, e_j^b(\phi)\delta_{ab} \\
(\mathscr{D}_\mu \eta)^a &= e_i^a\, (\mathscr{D}_\mu \eta)^i \\
R_{abcd} &= e_a^i e_b^j e_c^k e_d^l R_{ijkl} \\
\nabla_a \nabla_b V &= e_a^i e_b^j \nabla_i \nabla_j V
\end{aligned}
$$

$$
L = \frac{1}{2}\, (\mathscr{D}_\mu \eta)^a\, (\mathscr{D}^\mu \eta)^a - \frac{1}{2} R_{acbd}\, (D_\mu \phi)^c\, (D^\mu \phi)^d\, \eta^a \eta^b - \frac{1}{2}\, (\nabla_a \nabla_b V)\, \eta^a \eta^b
$$

- In 't Hooft form
- N_μ is automatically antisymmetric since the connection in the Cartan frame (spin-connection) is antisymmetric.
- Up and down indices are the same since metric is δ_{ab}.

$$L_{\text{c.t.}} = \frac{1}{16\pi^2\epsilon}\left[-\frac{1}{4}X_{ab}X^{ba} - \frac{1}{24}[Y_{\mu\nu}]_{ab}[Y_{\mu\nu}]^{ba}\right]$$

But all indices are contracted, so we have a scalar quantity which can be evaluated in any coordinate system:

$$X_{ab}X^{ba} = X_{ij}X^{ji}$$

so we do not actually have to make any transformations to the Cartan frame.

$$\begin{aligned} X_{ij} &= -R_{ikjl}\,(D_\mu\phi)^k\,(D^\mu\phi)^l - \left(\nabla_i\nabla_j V\right) \\ [Y_{\mu\nu}]^i{}_j &= R^i{}_{jkl}\,(D_\mu\phi)^k\,(D_\nu\phi)^l + F^\alpha_{\mu\nu}\,t^i_{\alpha;j}\,. \end{aligned}$$

Raise indices using g^{ij}.

Result holds for EFTs with operators up to two derivatives, but arbitrary high dimension.

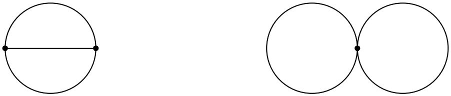
In a renormalizable theory, the kinetic term is canonical so $g_{ij} = \delta_{ij}$ and $R_{ijkl} = 0$.

If one includes operators of dim 6 or higher, have $R_{ijkl} \neq 0$. Then the one-loop correction induces four-derivative terms, two-loops induces 6 derivative terms, etc.

Two Loops

Jenkins, AM, Naterop, Pagès JHEP 12 (2023) 165; JHEP 02 (2025) 131

Two skeleton graphs with η^3 and η^4 vertices.



$$\mathcal{L} = A_{abc}\eta^a\eta^b\eta^c + A^{\mu}_{a|bc}(D_\mu\eta)^a\eta^b\eta^c + A^{\mu\nu}_{ab|c}(D_\mu\eta)^a(D_\nu\eta)^b\eta^c$$
$$+ B_{abcd}\eta^a\eta^b\eta^c\eta^d + B^{\mu}_{a|bcd}(D_\mu\eta)^a\eta^b\eta^c\eta^d + B^{\mu\nu}_{ab|cd}(D_\mu\eta)^a(D_\nu\eta)^b\eta^c\eta^d\,.$$

$A^{\mu}_{a|bc}(\text{completely symmetric}) = 0, \quad B^{\mu}_{a|bcd}(\text{completely symmetric}) = 0$

and $A^{\mu\nu}_{ab|c} \to 0$, which simplified the two loop computation.

$O(N)$ symmetry greatly simplifies result.

Two Loop Formula

66 terms:

$$\mathcal{L}_{\text{c.t.}} = \frac{1}{(16\pi^2)^2}\left[a_{2,1} A_{abc} A_{abd} X_{cd} + \ldots\right] \qquad a_{2,1} = \frac{9}{2\epsilon^2} - \frac{9}{2\epsilon}$$

Use the Riemann normal coordinate expansion to order η^4 and go to the Cartan frame:

- Tensors automatically have the correct symmetry properties using the symmetries of the Riemann curvature tensor including the Bianchi identity
- $A^{\mu\nu}_{ab|c} = 0$ automatically.
- All tensors in terms of R_{abcd}, its covariant derivative $\nabla_e R_{abcd}$ and covariant derivatives of the potential

Two Loop Formula

$$A_{abc} = -\frac{1}{6}\nabla_{(a}\nabla_b\nabla_{c)}V - \frac{1}{18}\left(\nabla_a R_{bdce} + \nabla_b R_{cdae} + \nabla_c R_{adbe}\right)(D_\mu\phi)^d(D^\mu\phi)^e$$

$$A^\mu_{a|bc} = \frac{1}{3}\left(R_{abcd} + R_{acbd}\right)(D^\mu\phi)^d$$

$$A^{\mu\nu}_{ab|c} = 0$$

$$B_{abcd} = -\frac{1}{24}\nabla_a\nabla_b\nabla_c\nabla_d V - \frac{1}{24}\nabla_a\nabla_d R_{becf}(D_\mu\phi)^e(D^\mu\phi)^f + \frac{1}{6}R_{eabf}R_{ecdg}(D_\mu\phi)^f(D^\mu\phi)^g \quad \text{sym}(abcd)$$

$$B^\mu_{a|bcd} = \frac{1}{4}\left(\nabla_d R_{abce}\right)(D^\mu\phi)^e \quad \text{sym}(bcd)$$

$$B^{\mu\nu}_{ab|cd} = -\frac{1}{12}\eta^{\mu\nu}\left(R_{acbd} + R_{adbc}\right)$$

Note that an expansion of $(\phi\cdot\phi)(\partial_\mu\phi\cdot\partial^\mu\phi)$ using $\phi\to\phi+\eta$ will generate $A^{\mu\nu}_{ab|c}$. So the geometric method is much simpler.

Can switch back to coordinate indices in the final expression.

One Loop Applications

χPT

$O(N)$ model, χPT, SMEFT.

The χPT Lagrangian at order p^2 is

$$\mathcal{L}_2 = \frac{f_\pi^2}{4} \langle u_\mu \, u^\mu \rangle \,, \qquad u_\mu = i(u^\dagger \, \partial_\mu u - u \, \partial_\mu u^\dagger) \,, \qquad u = e^{i\boldsymbol{\pi}/F}$$

The order p^4 Lagrangian is

$$\mathcal{L}_4 = \widehat{L}_0 \langle u_\mu u_\nu u^\mu u^\nu \rangle + \widehat{L}_1 \langle u \cdot u \rangle^2 + \widehat{L}_2 \langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + \widehat{L}_3 \left\langle (u \cdot u)^2 \right\rangle$$

The Riemann curvature tensor is

$$R_{abcd} = \frac{1}{f_\pi^2} f_{abg} f_{cdg} + \mathcal{O}(\pi^2)$$

χPT

The one-loop terms are order p^4, and we get the correct one-loop counterterms

$SU(n)$ — n is the number of flavors.

$$\widehat{L}_i = (c\mu)^{-2\epsilon}\left[-\frac{1}{2\epsilon}\frac{1}{16\pi^2}\widehat{\Gamma}_i + \widehat{L}_i^r(\mu)\right], \qquad c^2 = \frac{e^{\gamma_E - 1}}{4\pi}$$

$$\widehat{\Gamma}_0 = \frac{n}{48}, \qquad \widehat{\Gamma}_1 = \frac{1}{16}, \qquad \widehat{\Gamma}_2 = \frac{1}{8}, \qquad \widehat{\Gamma}_3 = \frac{n}{24},$$

Gasser, Leutwyler, Ann Phys 158 (1984) 142, NPB 250 (1985) 465

SMEFT Bosons Corrections

Helset, Jenkins, AM, JHEP 02 (2023) 063

Include SM $m_H^2, g_1, g_2, g_3, \lambda$

$$\begin{array}{ll} \text{dim } 6 & {}^6C_{H^6},\ {}^6C_{H^4\Box},\ {}^6C_{H^4D^2},\ {}^6C_{G^2H^2},\ {}^6C_{W^2H^2},\ {}^6C_{B^2H^2},\ {}^6C_{WBH^2}, \\ \text{dim } 8 & {}^8C_{H^8},\ {}^8C^{(1)}_{H^6D^2},\ {}^8C^{(2)}_{H^6D^2},\ {}^8C^{(1)}_{G^2H^4},\ {}^8C^{(1)}_{W^2H^4},\ {}^8C^{(3)}_{W^2H^4},\ {}^8C^{(1)}_{B^2H^4},\ {}^8C^{(1)}_{WBH^4}. \end{array}$$

Compute running of H^6, H^4D^2, X^2H^2 ; $X^4, H^8, H^6D^2, H^4D^4, X^2H^4, X^3H^2, X^2H^2D, XH^4D^2$.

$$\begin{array}{lcccc} \text{dim } 8 \propto & \text{dim } 8 & (\text{dim } 6)^2 & & \\ \text{dim } 6 \propto & \text{dim } 6 & m_H^2\,(\text{dim } 8) & m_H^2\,(\text{dim } 6)^2 & \\ \text{dim } 4 \propto & 1 & m_H^2\,(\text{dim } 6) & m_H^4\,(\text{dim } 8) & m_H^4\,(\text{dim } 6)^2 \\ \text{dim } 2 \propto & m_H^2 & m_H^4\,(\text{dim } 6) & m_H^6\,(\text{dim } 8) & m_H^6\,(\text{dim } 6)^2 \end{array}$$

where each term has a coefficient of g_1, g_2, g_3, λ

agrees for terms common to both computations with

Das Bakshi, Chala, Díaz-Carmona, Guedes, EPJ+ 137 (2022) 973

Das Bakshi, Díaz-Carmona, JHEP 06 (2023) 123

SMEFT Fermions

Alonso, Kanshin, Saa, PRD 97 (2018) 035010

Finn, Karamitsos, Pilaftsis, EPJC 81 (2021) 572

Gattus, Pilaftsis, 2307.01126

SMEFT to dim 8 (bosonic terms from fermion loops)

Assi, Helset, AM, Pagès, Shen, 2307.03187

common terms agree with

Das Bakshi, Chala, Díaz-Carmona, Guedes, EPJ+ 137 (2022) 973

Das Bakshi, Díaz-Carmona, JHEP 06 (2023) 123

Supergeometry for two-fermion operators:

Assi, Helset, Pagès, Shen, arXiv:2504.18537

Two Loop Applications

$O(n)$ EFT including dim 6 terms

Cao, Herzog, Melia, Nepveu, JHEP 09 (2021) 014

Dim 6 real scalar (n=1) to 5 loops, and dim 6 complex scalar (n=2) to 4 loops.

$$L = \frac{1}{2}(\partial_\mu \phi \cdot \partial_\mu \phi) - \frac{1}{2} m^2 (\phi \cdot \phi) - \frac{1}{4}\lambda(\phi \cdot \phi)^2 \\ + C_{\phi^6}(\phi \cdot \phi)^3 + C_E(\phi \cdot \phi)(\partial_\mu \phi \cdot \partial_\mu \phi) - C_{\phi\Box}\partial_\mu(\phi \cdot \phi)\partial^\mu(\phi \cdot \phi)$$

Computed two loop RGE for the theory for arbitrary n

γ_ϕ is infinite

$$\gamma_\phi = \left\{-4(n-1)m^2 C_E\right\}_1 + \frac{1}{\epsilon}\left\{4(n-1)(n+2)\lambda m^2 C_E\right\}_2 \\ + \left\{(n+2)\lambda^2 - \frac{8}{3}(n+2)\lambda m^2 C_E\right\}_2$$

$\{\}_1$ has $1/(16\pi^2)$ and $\{\}_2$ has $1/(16\pi^2)^2$

χPT

Bijnens, Colangelo, Ecker, Ann Phys 280 (2000) 100.

The p^6 Lagrangian including external sources is

$$\mathcal{L}_6 = \sum_{i=1}^{115} K_i Y_i$$

19 independent operators when external sources turned off.

Weinberg: p^6 from two-loop of order p^2 Lagrangian, or one-loop with one insertion of p^4 Lagrangian.

$$K_i = \frac{(c\mu)^{-4\epsilon}}{f^2}\left[-\frac{1}{4\epsilon^2}\frac{1}{(16\pi^2)^2}\widehat{\Gamma}_i^{(2)} + \frac{1}{2\epsilon}\frac{1}{16\pi^2}\widehat{\Gamma}_i^{(1)} + \frac{1}{2\epsilon}\frac{1}{16\pi^2}\widehat{\Gamma}_i^{(L)} + K_i^r\right]$$

$\widehat{\Gamma}_i^{(1,2)}$ counterterms for two-loop graphs from the p^2 Lagrangian
$\widehat{\Gamma}_i^{(L)}$ counterterms for one-loop graphs with an insertion of $\mathcal{L}_4$.

χPT

Y	$\widehat{\Gamma}_i^{(2)}$	$16\pi^2\widehat{\Gamma}_i^{(1)}$	$\widehat{\Gamma}_i^{(L)}$
1	$\frac{5}{48}+\frac{1}{64}n^2$	$-\frac{67}{576}-\frac{7}{1728}n^2$	$\frac{1}{12}n\widehat{L}_0+3\widehat{L}_1+\frac{1}{6}\widehat{L}_2+\frac{17}{24}n\widehat{L}_3$
2	$\frac{1}{576}n$	$-\frac{31}{6912}n$	$\frac{5}{24}\widehat{L}_0^r-\frac{1}{24}n\widehat{L}_2^r+\frac{5}{48}\widehat{L}_3^r$
3	$-\frac{5}{48}+\frac{1}{2304}n^2$	$\frac{61}{2304}+\frac{11}{27648}n^2$	$\frac{1}{48}n\widehat{L}_0-\frac{7}{6}\widehat{L}_1-\frac{13}{12}\widehat{L}_2+\frac{1}{96}n\widehat{L}_3$
4	$-\frac{11}{72}n$	$-\frac{49}{3456}n$	$-\frac{5}{4}\widehat{L}_0^r-2n\widehat{L}_1^r-\frac{3}{4}n\widehat{L}_2^r-\frac{35}{24}\widehat{L}_3^r$
5	$-\frac{1}{768}n^2$	$-\frac{23}{256}-\frac{49}{27648}n^2$	$-\frac{11}{48}n\widehat{L}_0+\frac{11}{6}\widehat{L}_1-\frac{11}{12}\widehat{L}_2+\frac{5}{96}n\widehat{L}_3$
6	$-\frac{13}{32}n$	$-\frac{5}{192}n$	$-\frac{3}{4}\widehat{L}_0^r-6n\widehat{L}_1^r-\frac{9}{4}n\widehat{L}_2^r-\frac{27}{8}\widehat{L}_3^r$
49	$-\frac{5}{576}n^2$	$\frac{5}{48}+\frac{1}{2304}n^2$	$\frac{1}{4}n\widehat{L}_0-\frac{4}{3}\widehat{L}_1+\frac{2}{3}\widehat{L}_2-\frac{13}{24}n\widehat{L}_3$
50	$\frac{1}{32}n$	$\frac{5}{128}n$	$-\frac{1}{4}\widehat{L}_0^r+\frac{1}{4}n\widehat{L}_2^r+\frac{7}{8}\widehat{L}_3^r$
51	$\frac{1}{64}$	$\frac{5}{256}$	$\frac{1}{4}\widehat{L}_2^r$
52	$-\frac{11}{384}n^2$	$\frac{17}{128}+\frac{77}{13824}n^2$	$-\frac{17}{24}n\widehat{L}_0-\frac{1}{3}\widehat{L}_1+\frac{1}{6}\widehat{L}_2-\frac{49}{48}n\widehat{L}_3$
53	$-\frac{1}{64}n$	$-\frac{5}{256}n$	$\widehat{L}_0^r-\frac{5}{4}\widehat{L}_3^r$
54	$\frac{1}{48}n^2$	$-\frac{17}{64}-\frac{13}{3456}n^2$	$-\frac{1}{3}n\widehat{L}_0+\frac{2}{3}\widehat{L}_1-\frac{1}{3}\widehat{L}_2+\frac{7}{6}n\widehat{L}_3$
55	$-\frac{1}{24}n$	$-\frac{1}{72}n$	$-\frac{2}{3}\widehat{L}_0^r-\frac{2}{3}n\widehat{L}_2^r+\frac{1}{3}\widehat{L}_3^r$
56	$-\frac{1}{32}$	$\frac{3}{128}$	$-\frac{1}{2}\widehat{L}_2^r$
58	$\frac{1}{1152}n^2$	$\frac{11}{384}-\frac{13}{13824}n^2$	$\frac{1}{24}n\widehat{L}_0+\widehat{L}_1-\frac{1}{2}\widehat{L}_2+\frac{1}{48}n\widehat{L}_3$
59	$-\frac{1}{192}n$	$\frac{65}{2304}n$	$\widehat{L}_0^r-\frac{3}{4}\widehat{L}_3^r$
61	$\frac{7}{192}n$	$-\frac{23}{2304}n$	$\widehat{L}_0^r+\frac{5}{4}\widehat{L}_3^r$
62	$-\frac{1}{12}n$	$-\frac{5}{288}n$	$-\frac{13}{3}\widehat{L}_0^r-\frac{1}{3}n\widehat{L}_2^r-\frac{5}{6}\widehat{L}_3^r$
63	$-\frac{1}{8}$	$-\frac{1}{32}$	$-2\widehat{L}_2^r$

SMEFT

We include the SM Higgs sector, as well as insertions of the dimension six operators

$$C_H,\ C_{H\Box},\ C_{HD},\ C_{HG},\ C_{HW},\ C_{HB},\ C_{HWB},\ C_{H\widetilde{G}},\ C_{H\widetilde{W}},\ C_{H\widetilde{B}},\ C_{H\widetilde{W}B}\,.$$

and compute their contribution to anomalous dimensions from internal scalar loops.

$$\dot{C}_H = \left\{-3444\lambda^2 C_H + 7968\lambda^3 C_{H\Box} - 1992\lambda^3 C_{HD}\right\}_2\,,$$

Some very large coefficients

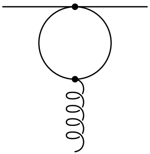
The Higgs field anomalous dimension is divergent at two loops, i.e. the 't Hooft consistency conditions are not satisfied,

$$\gamma_H = \left\{-\frac{1}{\epsilon}18\lambda m_H^2 C_{H\Box} + 6\lambda^2 - 8\lambda m_H^2 C_{H\Box} + 2\lambda m_H^2 C_{HD}\right\}_2$$

Infinite Anomalous Dimensions

Field Redefinitions in EFTs

EFTs use field redefinitions to remove redundant operators.



The penguin graph is divergent, and requires the counterterm

$$\mathrm{L} = \frac{4G_F}{\sqrt{2}}\frac{c_P}{\epsilon} g(\overline{\psi}\gamma^\mu T^A \psi)\,(D^\nu F_{\mu\nu})^A \ .$$

Make a field redefinition

$$A_\mu^A \to A_\mu^A - \frac{4G_F}{\sqrt{2}}\frac{c_P}{\epsilon} g\overline{\psi}\gamma^\mu T^A \psi\,,$$

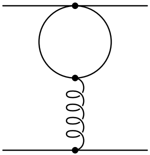
and replace it by a four-quark operator

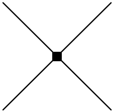
$$\mathrm{L} \to \frac{4G_F}{\sqrt{2}}\frac{c_P}{\epsilon} g(\overline{\psi}\gamma^\mu T^A \psi) g(\overline{\psi}\gamma_\mu T^A \psi)\,,$$

Field Redefinitions in EFTs

$$A_\mu^A \to A_\mu^A - \frac{4G_F}{\sqrt{2}} \frac{c_P}{\epsilon} g \overline{\psi} \gamma^\mu T^A \psi \,,$$

- A field redefinition with an infinite coefficient.
- Green's functions using the redefined Lagrangian are infinite, but the *S*-matrix is finite.
- There is no counterterm to cancel the penguin graph divergence, but the on-shell four-quark amplitude gets both the penguin and counterterm contributions and is finite.





Conclusions

- Geometrical methods provide a better understanding of the structure of QFT
- Help understand the experimental limits in a model-independent way
- Simplify the computation of radiative corrections
- Still a lot to understand
  - only looked at very limited field redefinitions
  - Loop diagrams for scattering amplitudes
  - Higher derivative terms