Three-body systems *from lattice QCD to Efimov*

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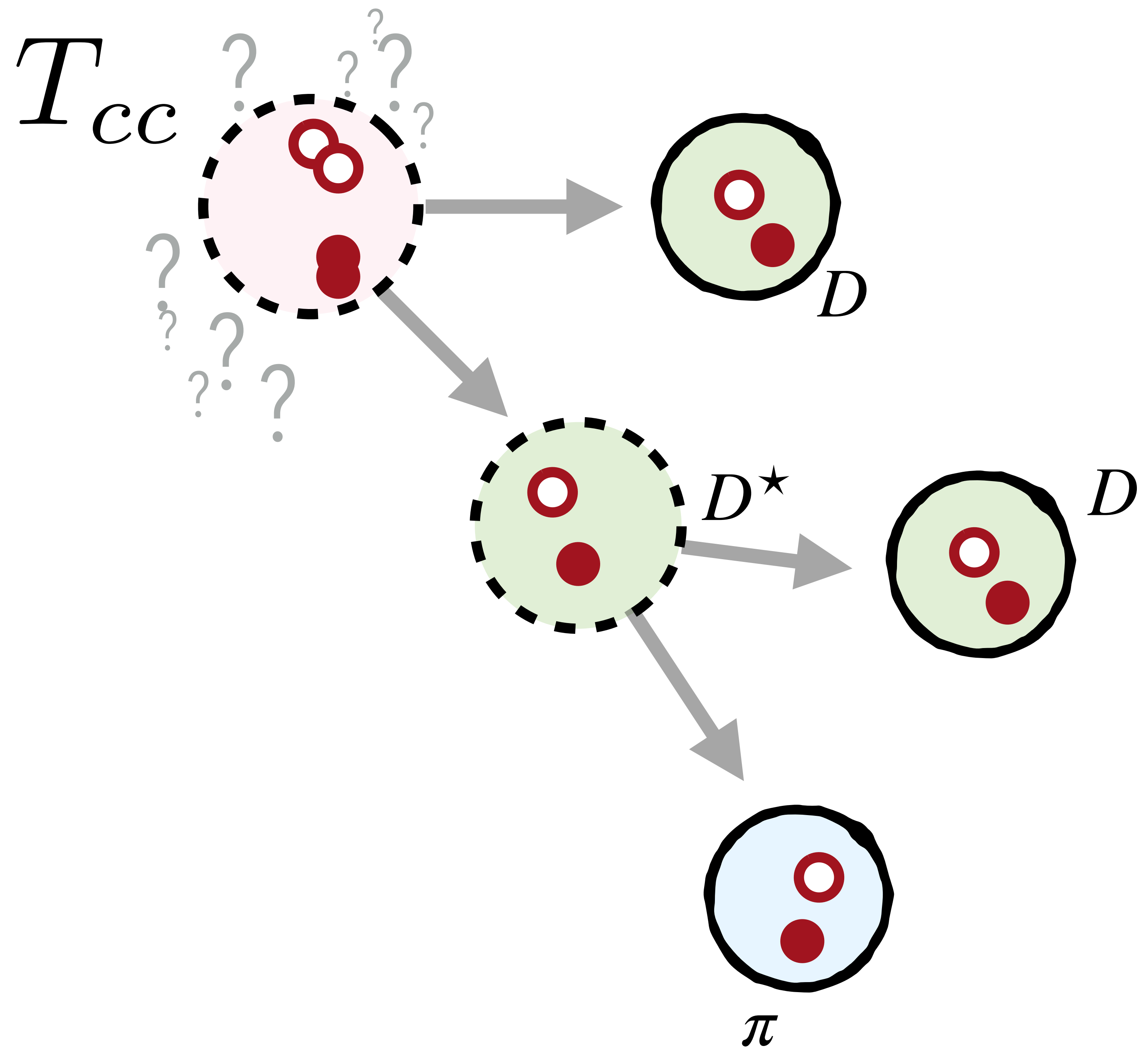
The "state" of my mind

$|$  $\rangle = a|\mathrm{NT}\rangle + b|\mathrm{PT}_a\rangle + c|\mathrm{AMOT}\rangle$

$$1 \sim |a| > |b| \gg |c|$$

why three-body systems?

- hadron spectroscopy



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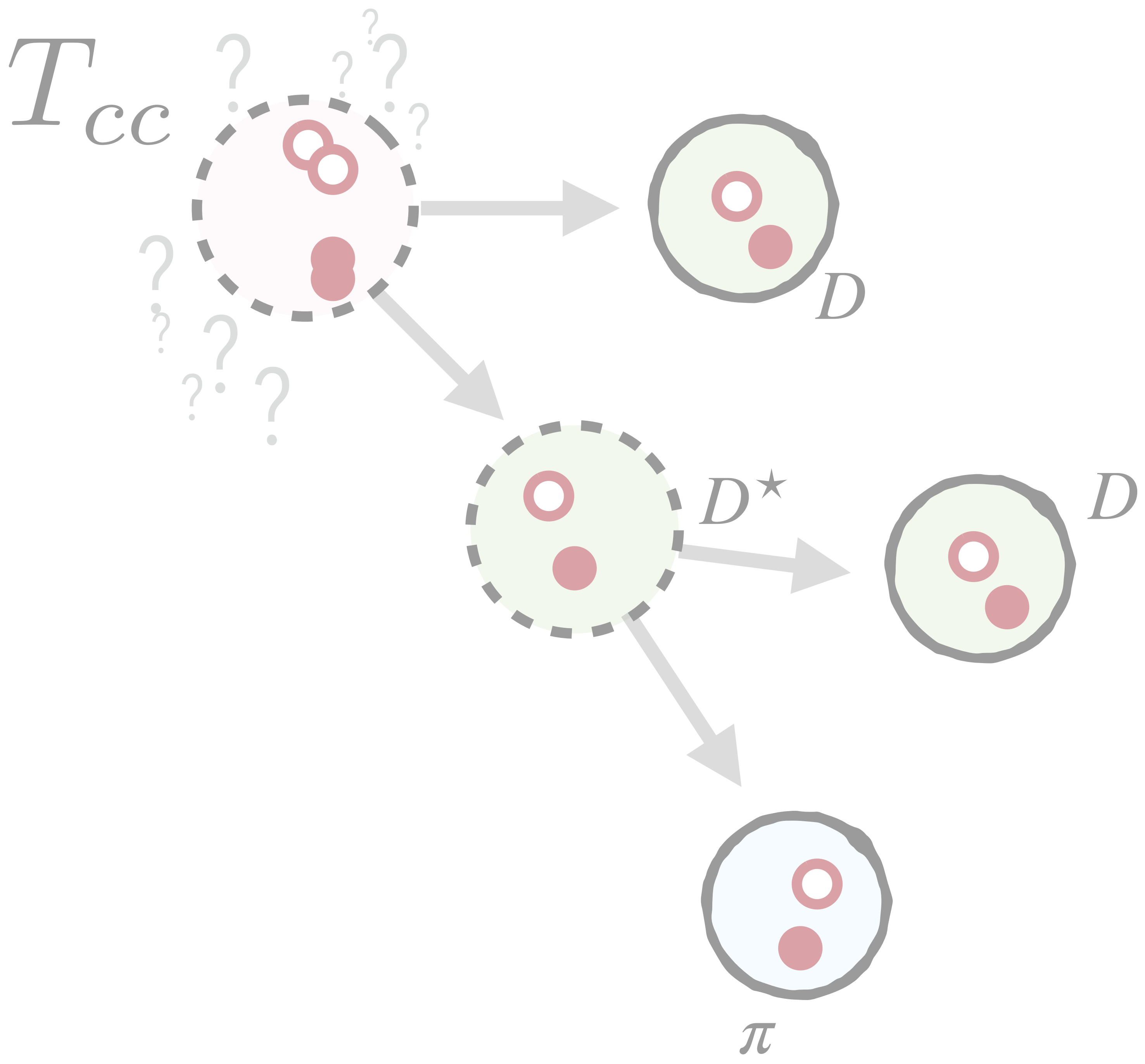
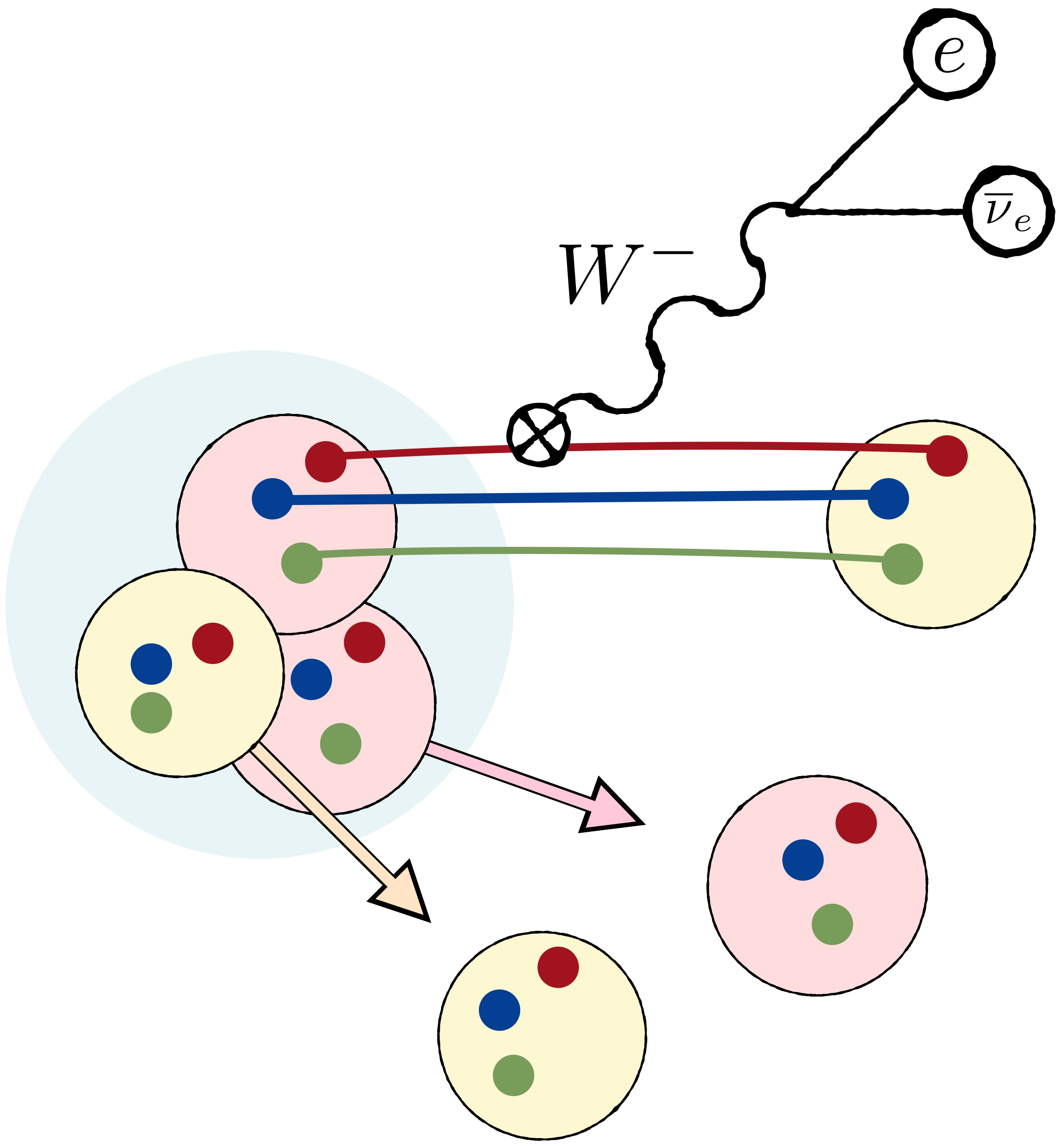
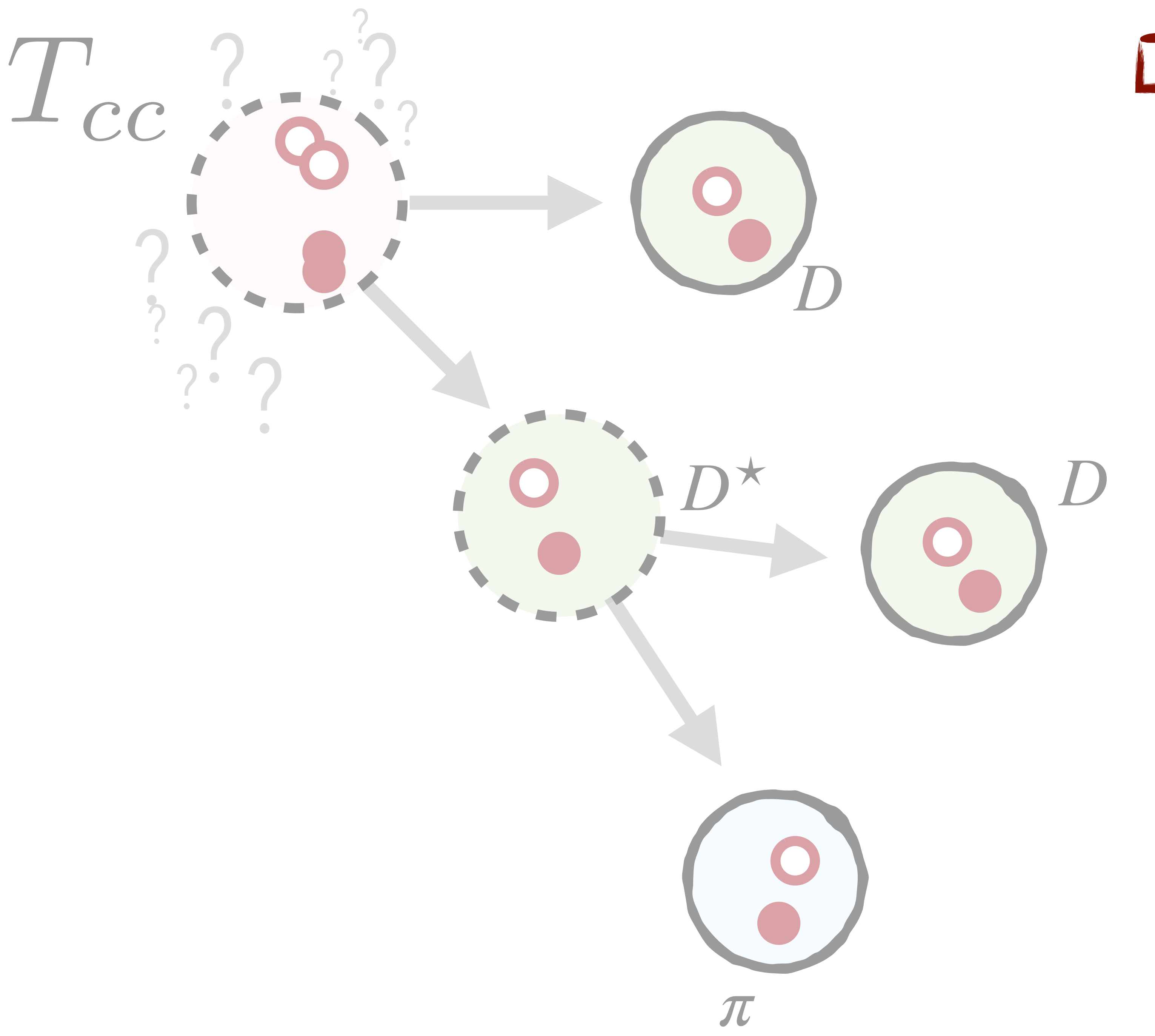


- nuclear structure / neutrino physics



why three-body systems?

- hadron spectroscopy



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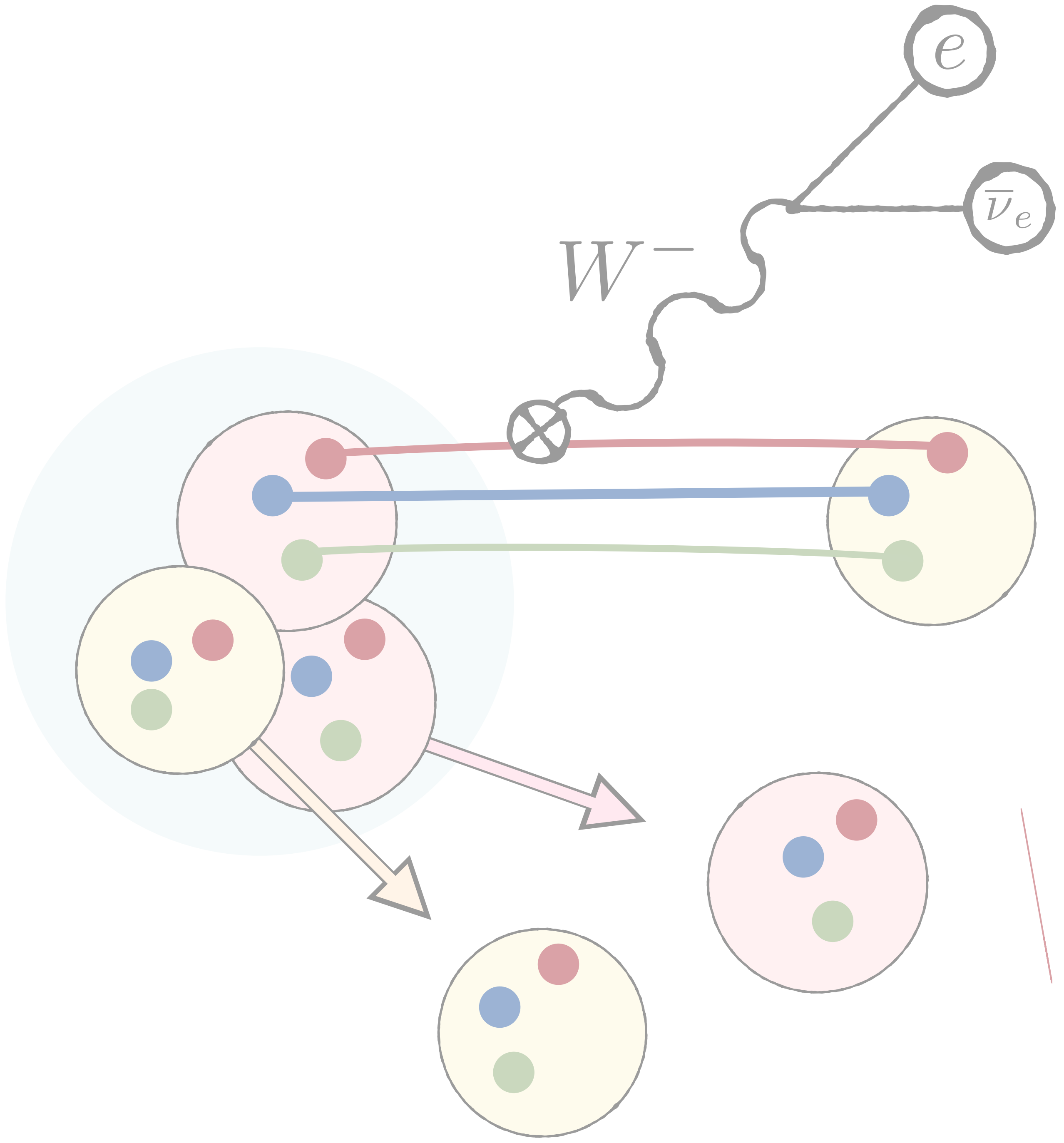
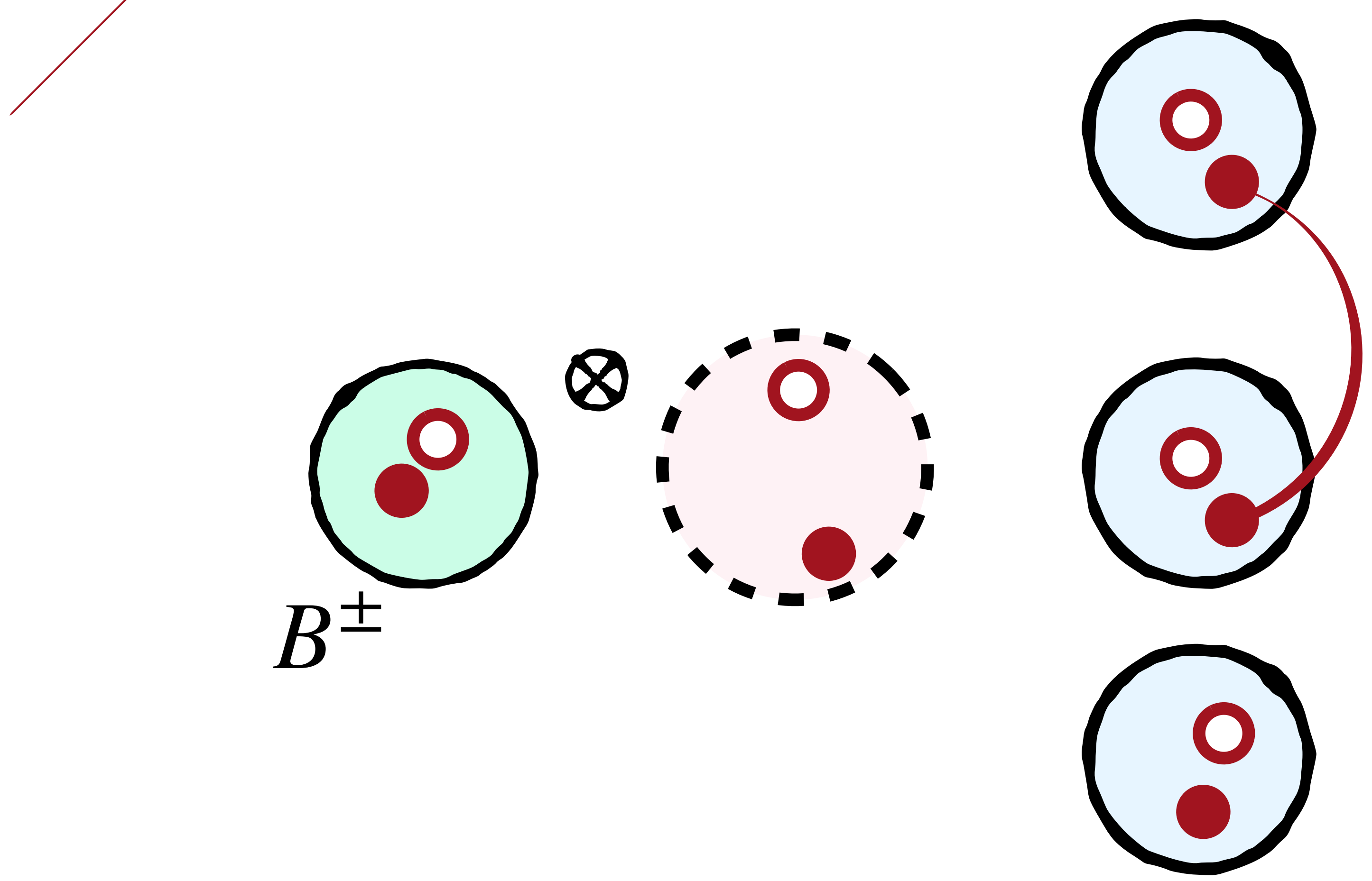
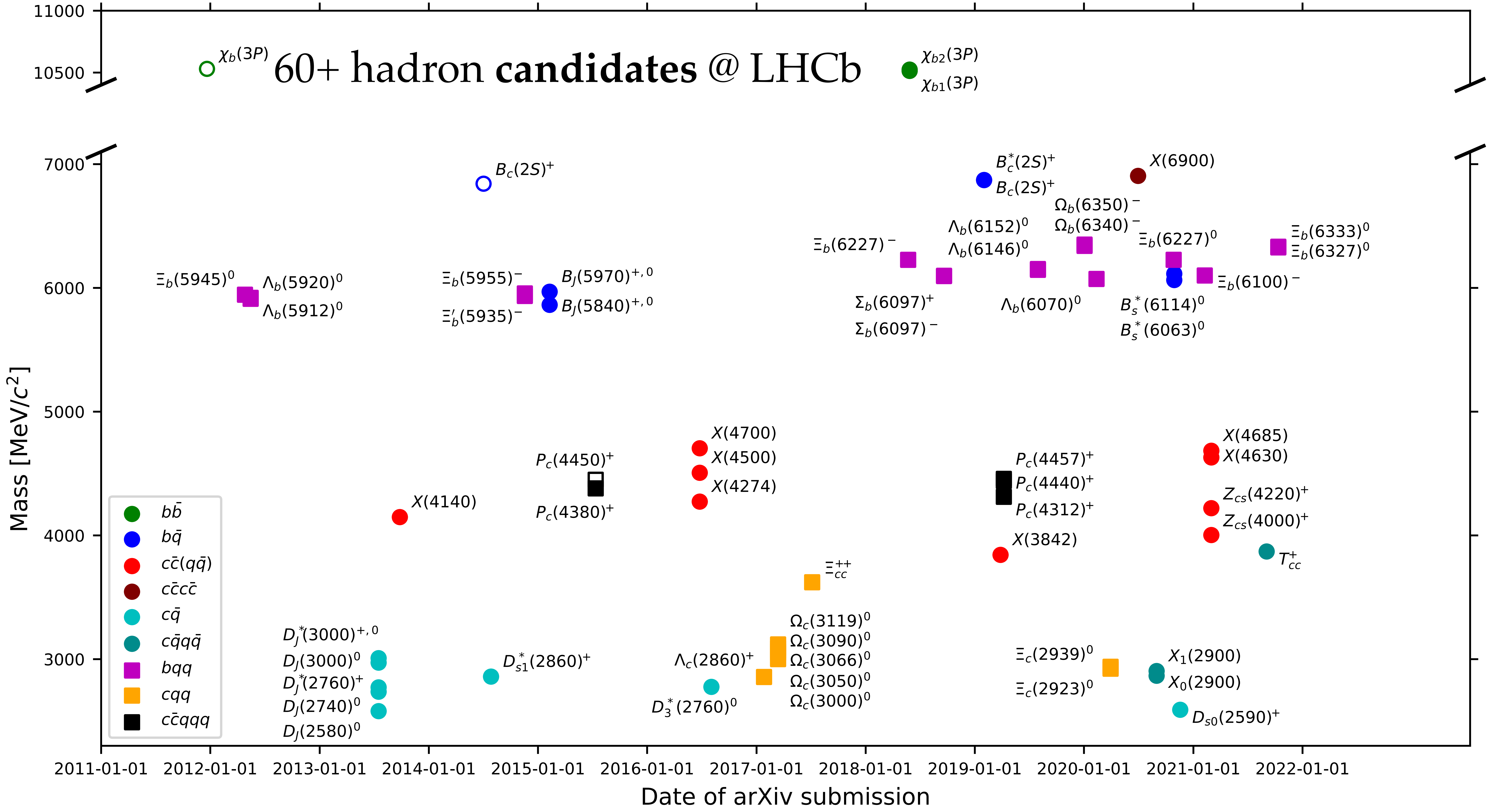


- precision tests

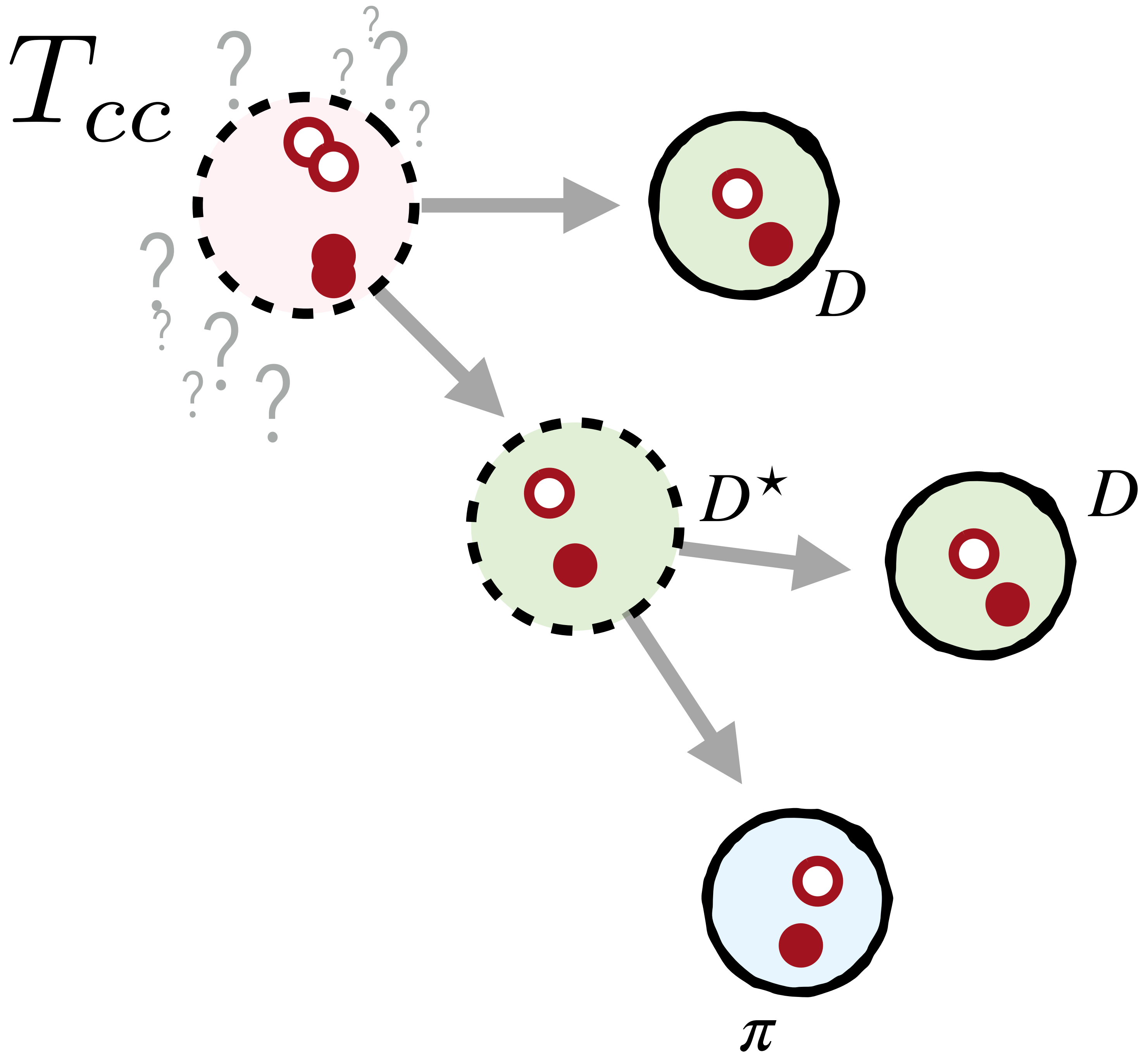


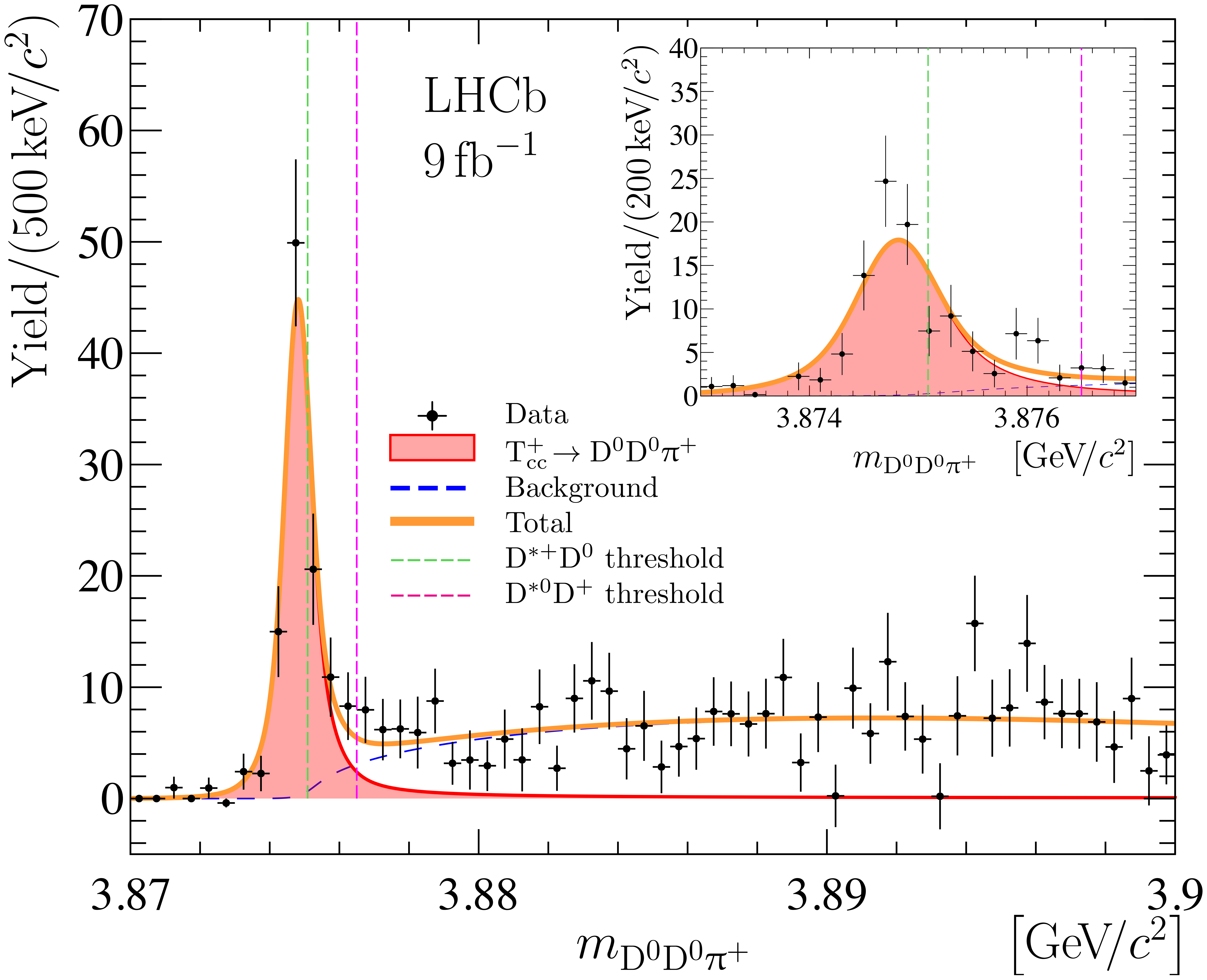
The particle zoo the remake



Numerous other experimental searches…

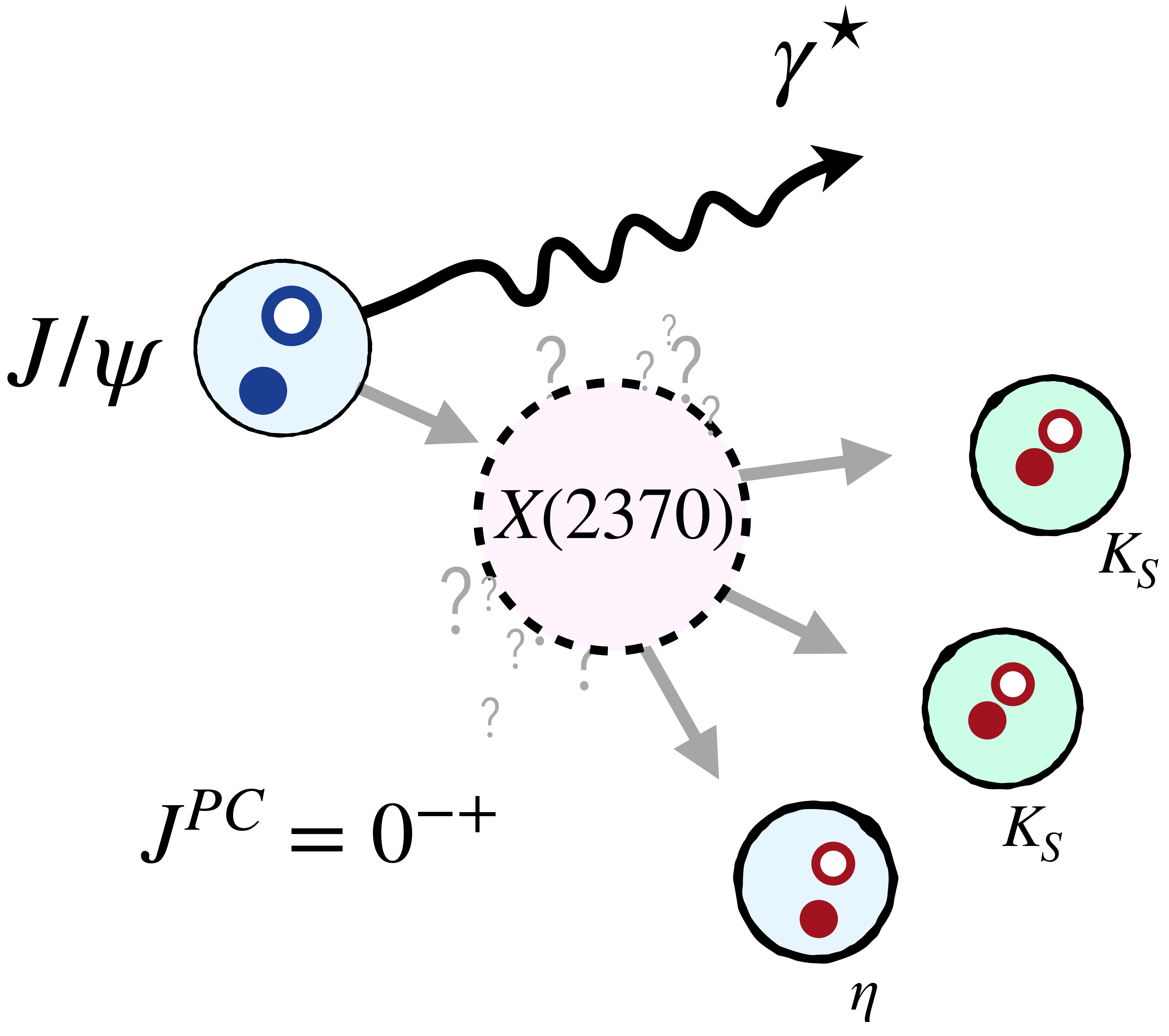
Tetraquarks?

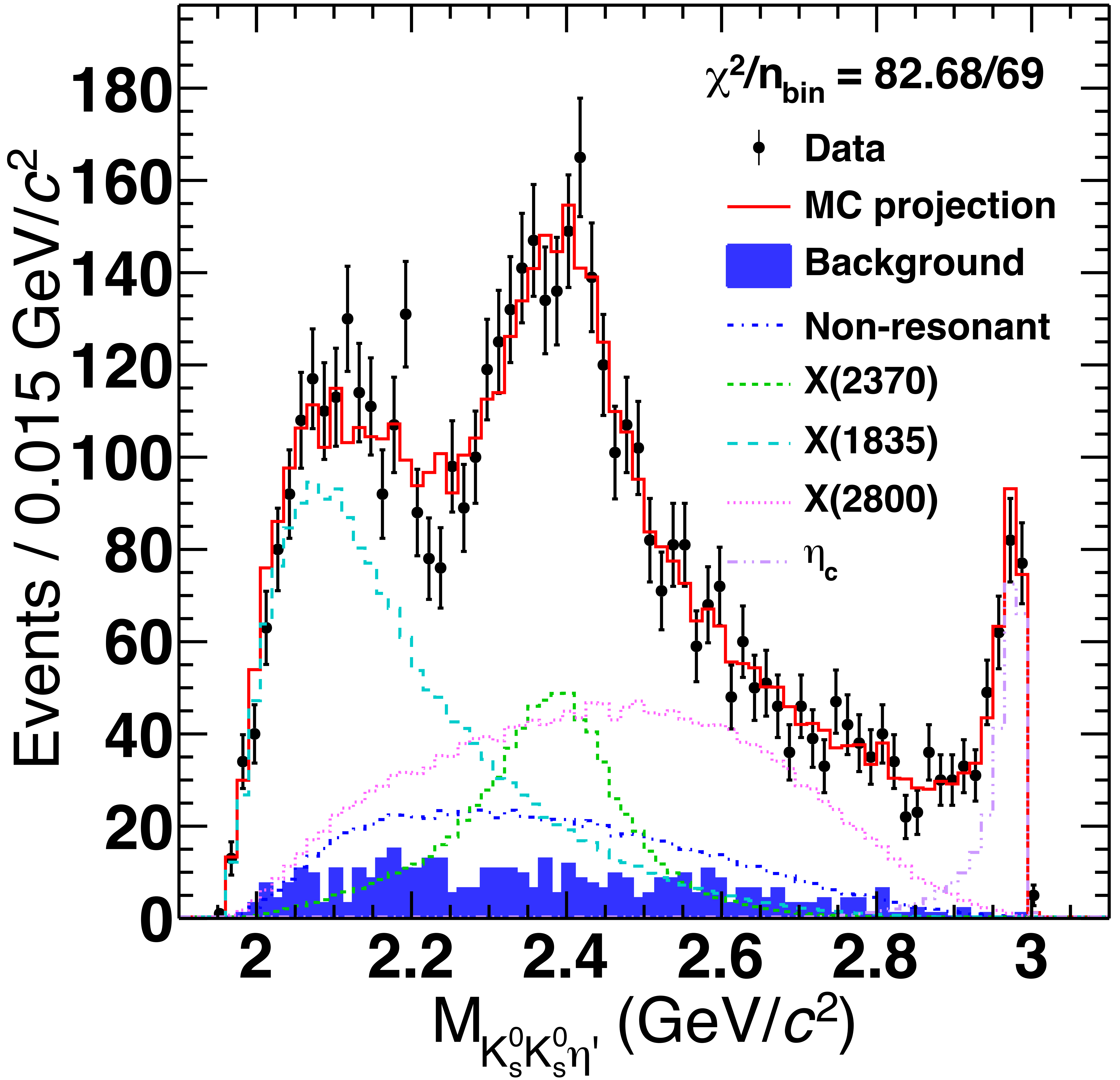




(2021)

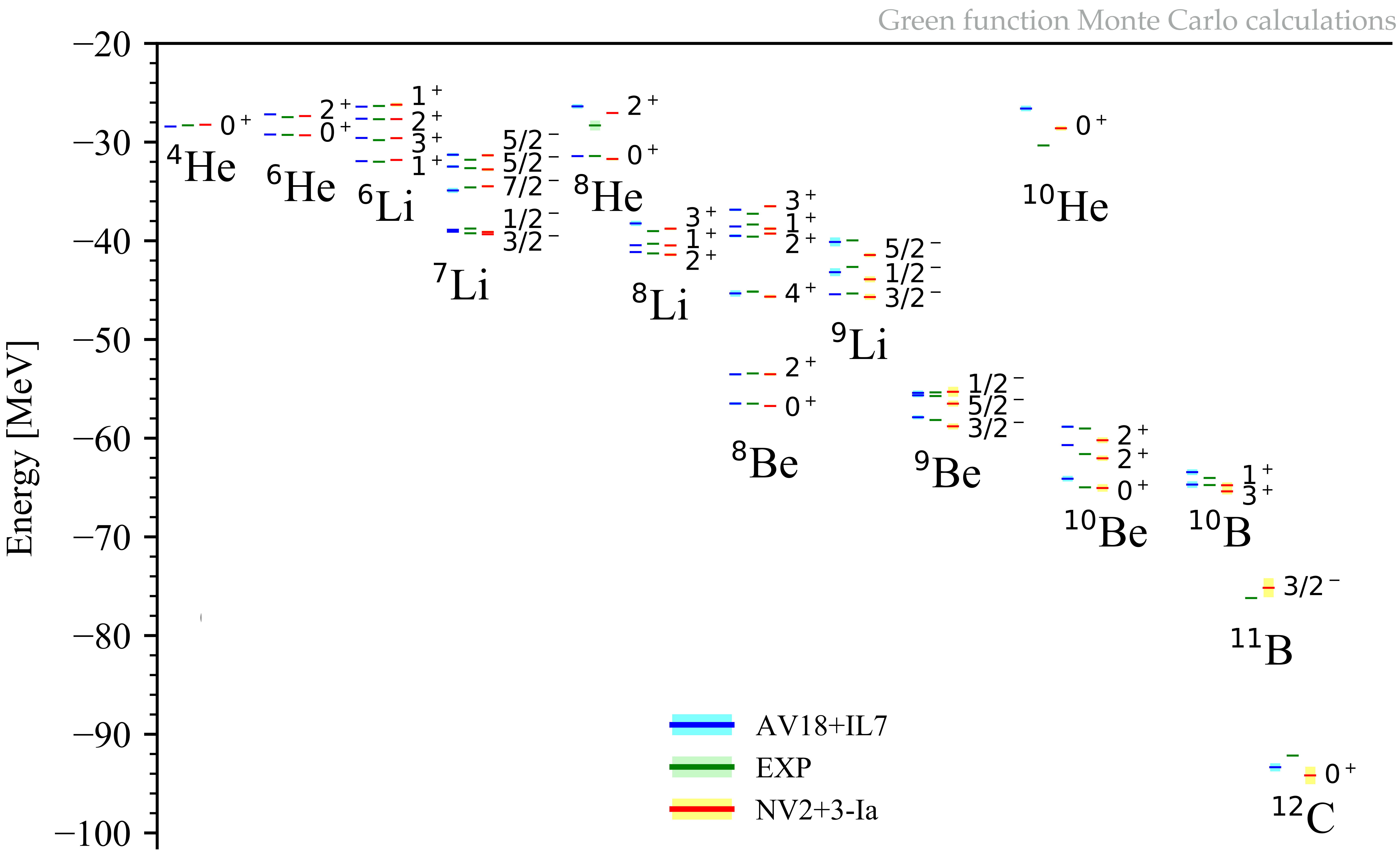
Glueballs?





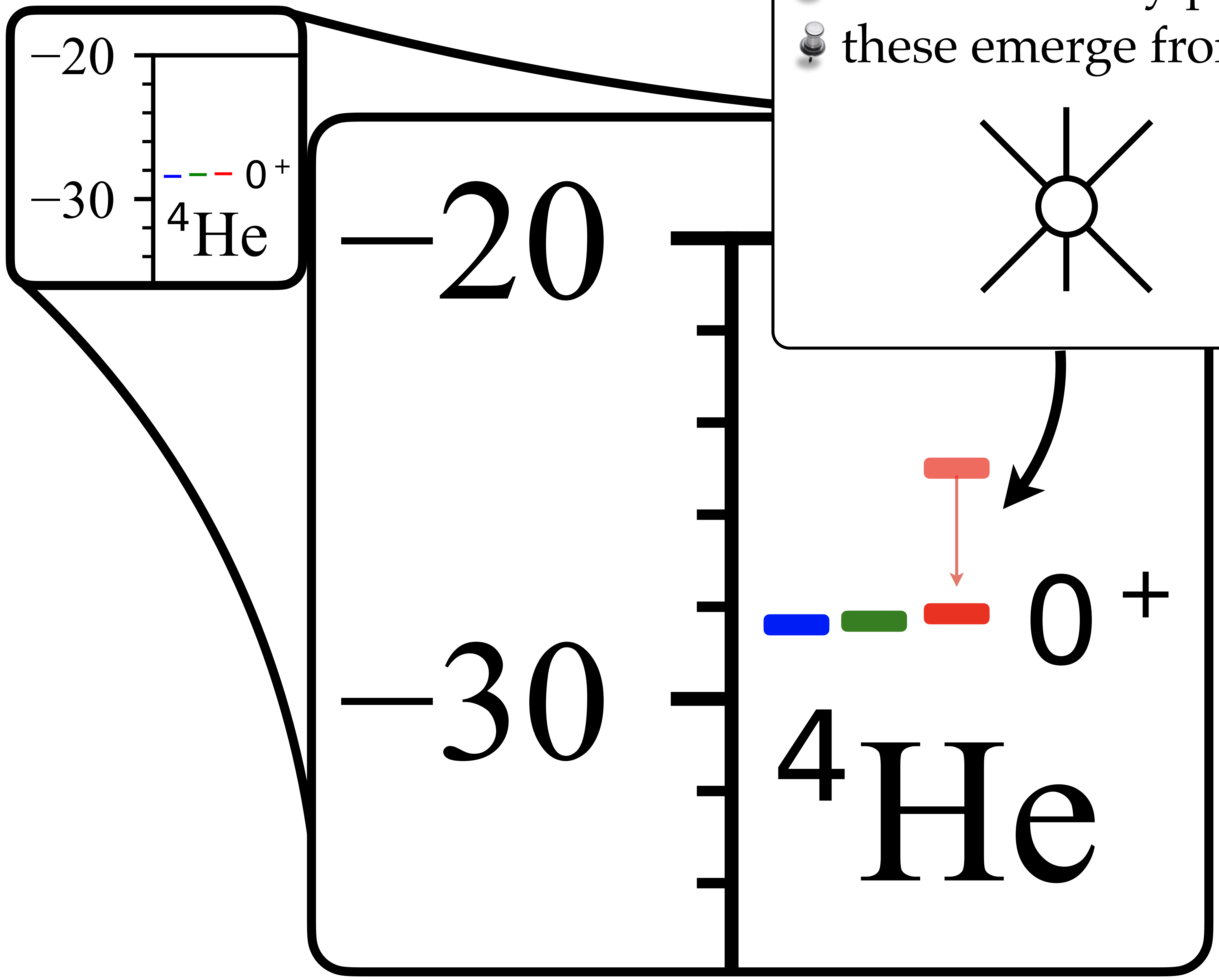
BESIII

Nuclear physics & precision tests

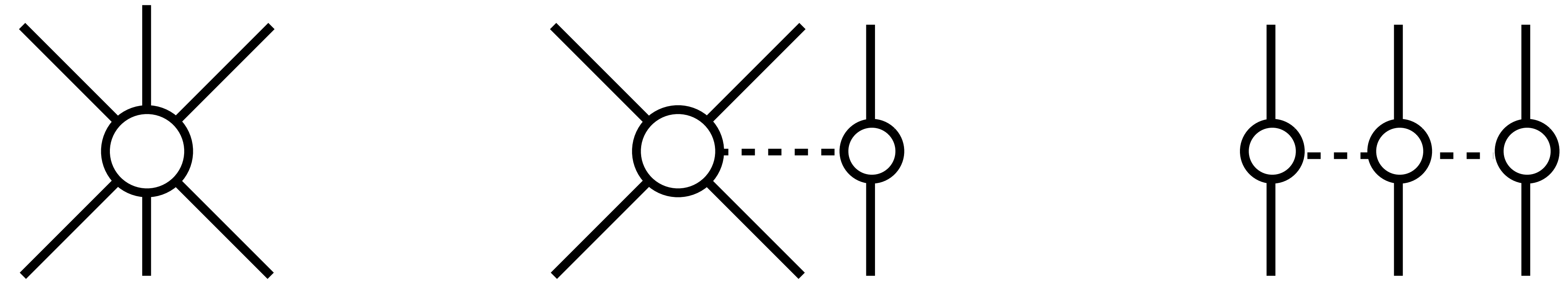


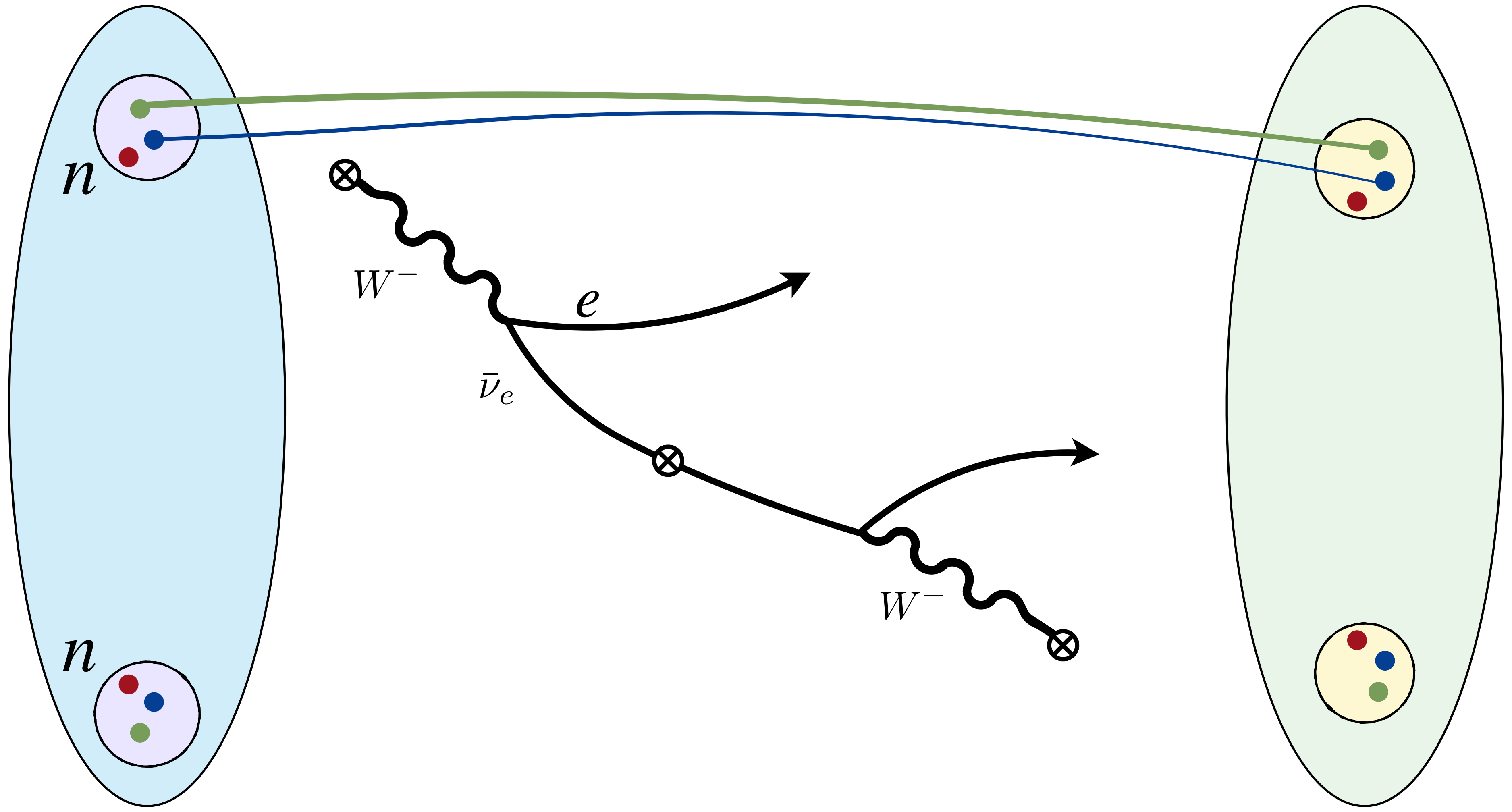
Piarulli, Baroni, Schiavilla et al. (2017)

Nuclear physics



- the three-body potentials shifts spectra by about 10%-20%
- these emerge from "local" & "non-local" interactions





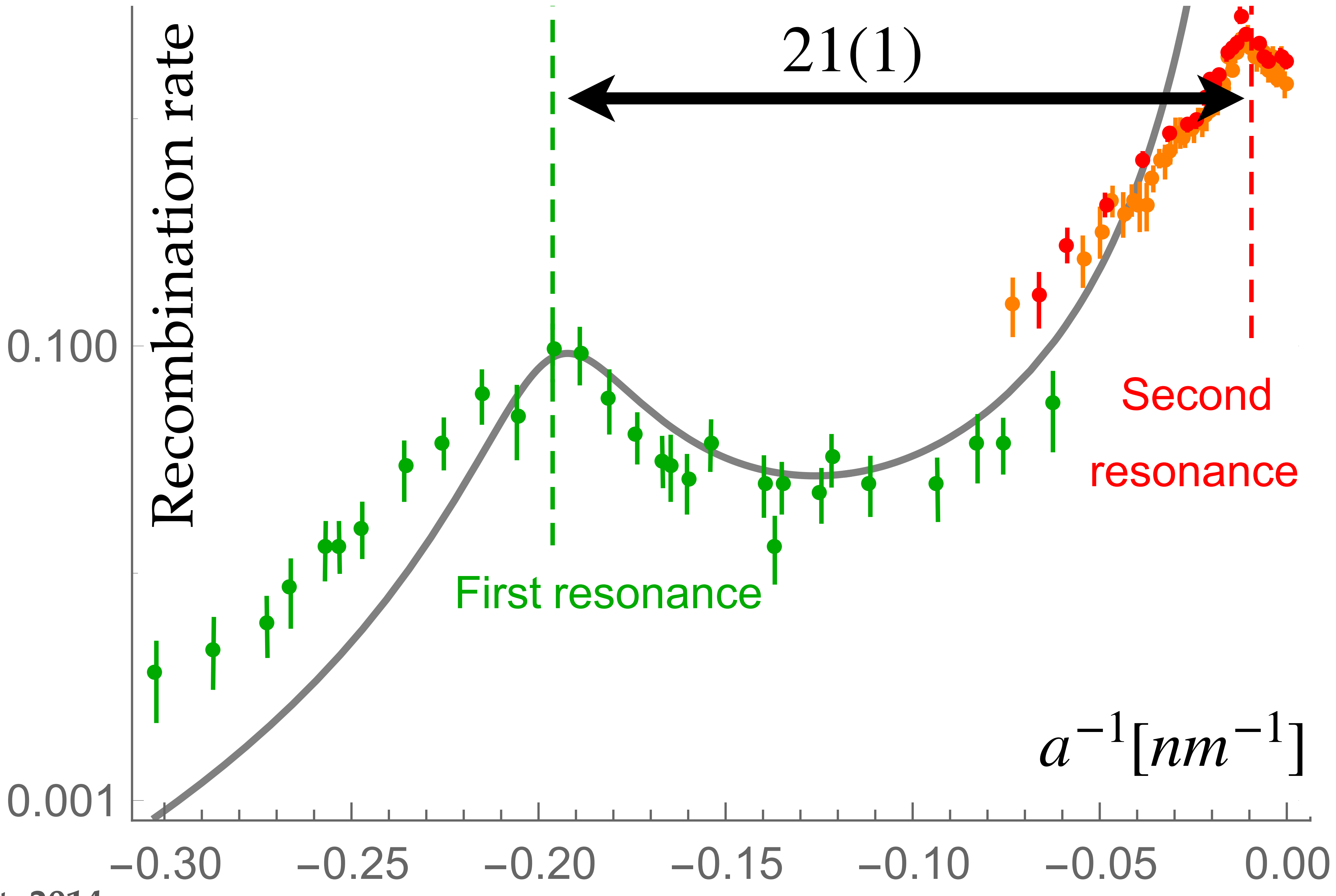
neutrionless double-beta decay

Efimov physics

Unitary limit: $p \cot \delta = -\frac{1}{a} + \frac{rp^2}{2} + \cdots = 0$

Pole in the two-body scattering amplitude at threshold: $\mathcal{M} \sim \frac{1}{p \cot \delta - ip} = \frac{1}{ip}$

Infinite tower of geometrically-separated three-body bound states: $E_{N+1} = E_N/\lambda^2$ where $\lambda = 22.69438$



Phys. Rev. Lett. **2014**

Vitaly Efimov

Overarching goal

non-perturbatively constrain two- and three-hadron scattering amplitudes directly from the standard model (including electroweak & BSM probes)

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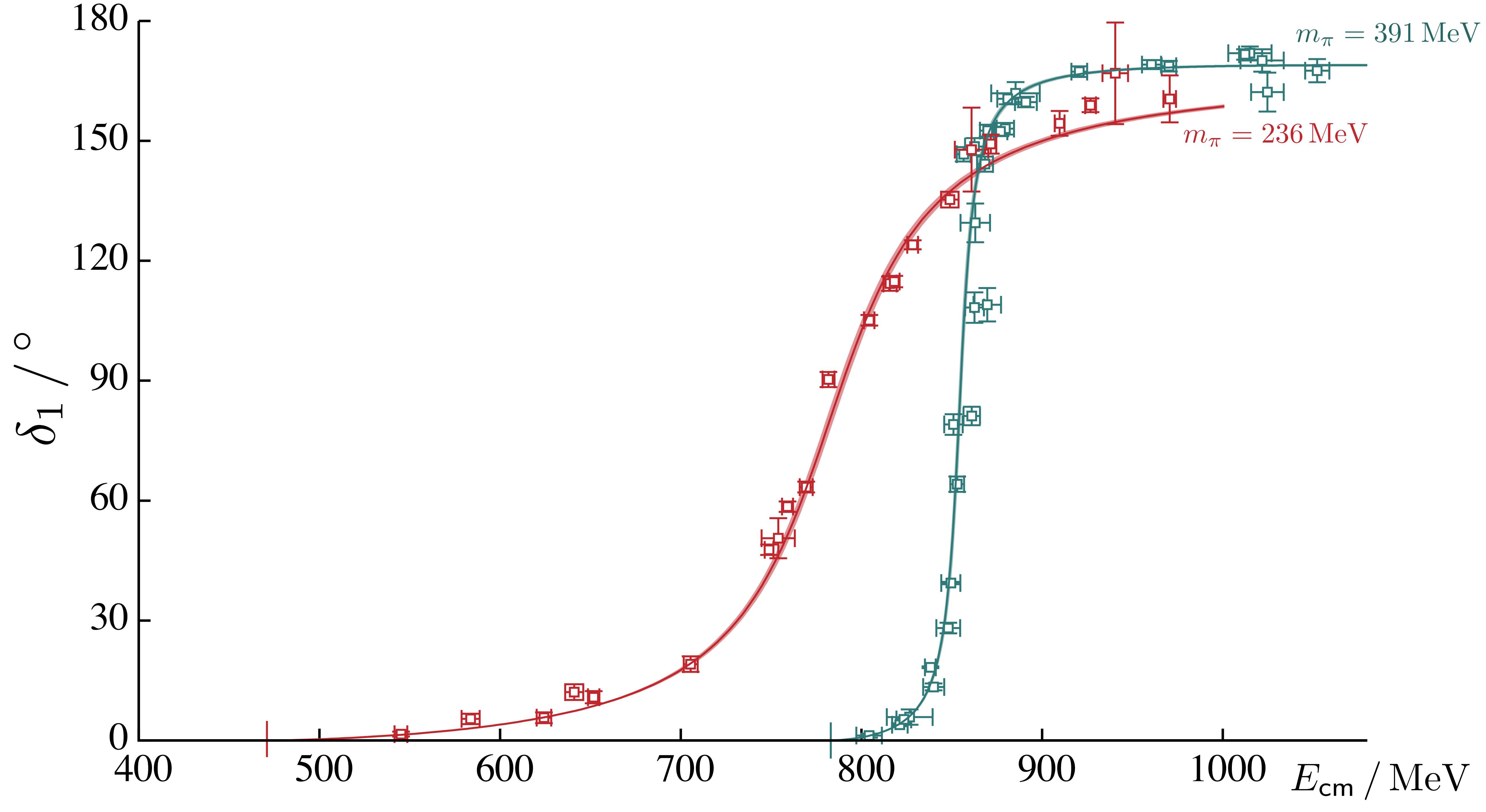
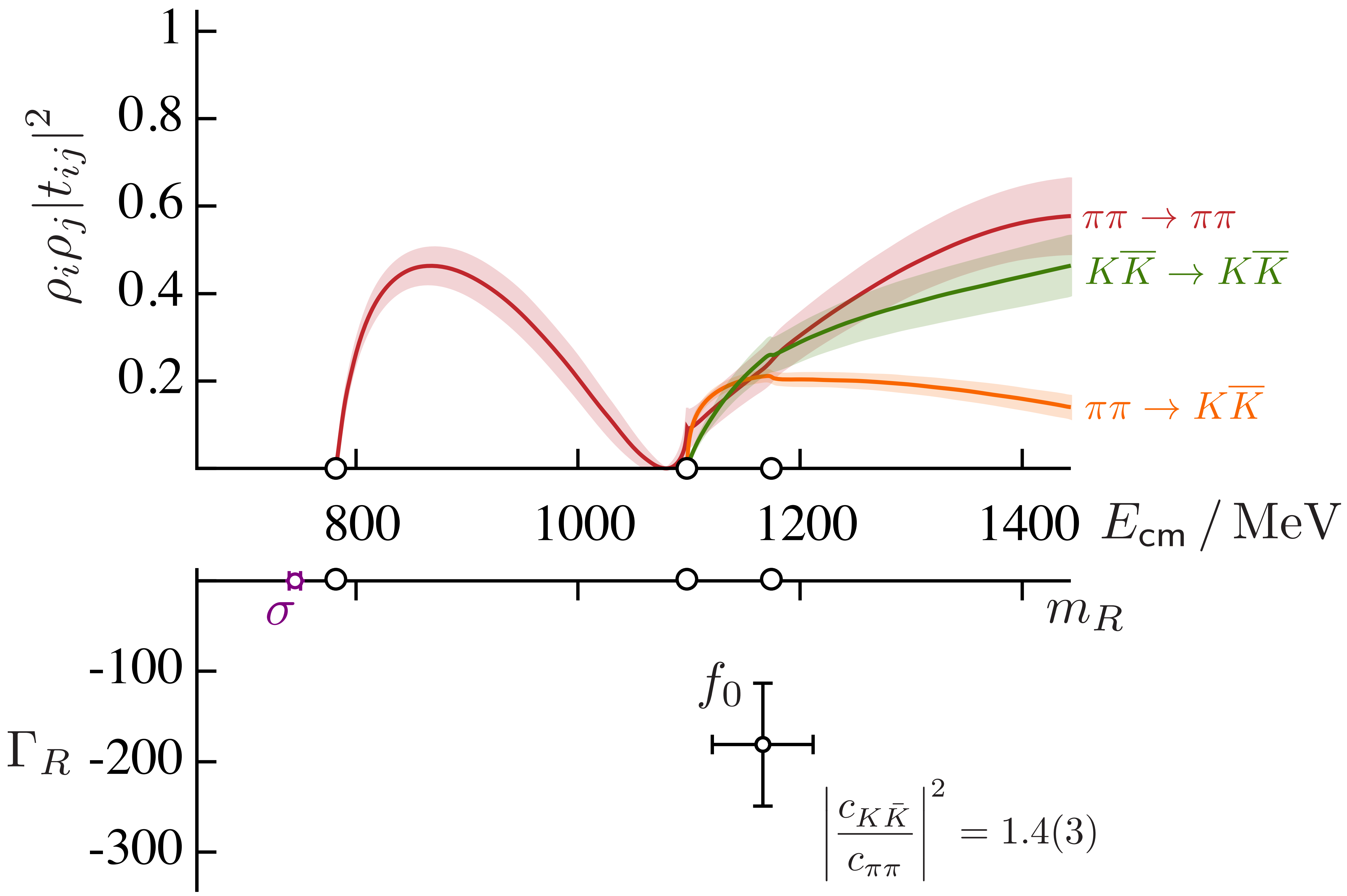
Two-body systems are well studied via lattice QCD





outline

S. R. Costa   Jackura   Dawid   Islam   Thomas

- ☐ integral equations
- ☐ angular momentum projection
- ☐ finite-volume formalism
- ☐ a lattice QCD calculation
- ☐ toy model calculations
- ☐ Efimov physics
- ☐ consistency checks and the breakdown of Lüscher

[won't present, but happy to discuss]

Hansen, RB, Edwards, Thomas, & Wilson (2020)
Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)
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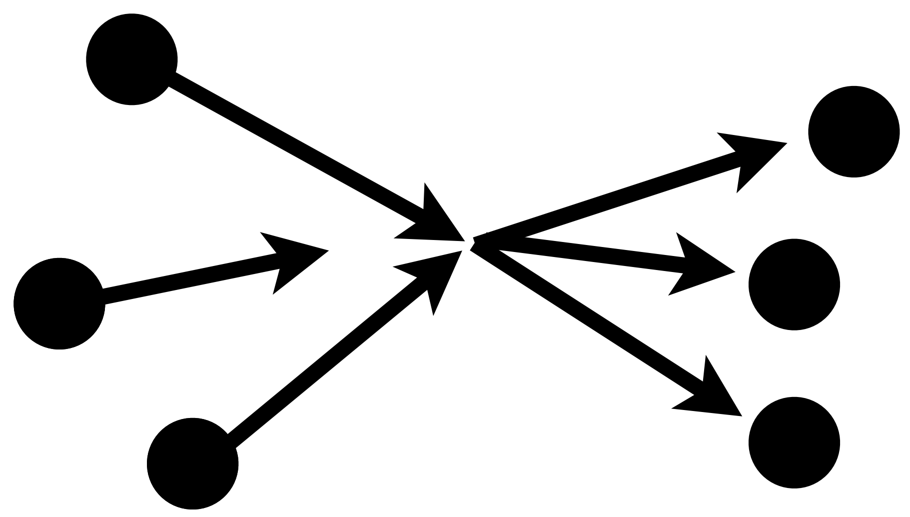
RB, S. R. Costa, Jackura, (2024)
Dawid, RB, Islam, Jackura, (2023)

Arsenal of non-perturbative tools

Scattering theory

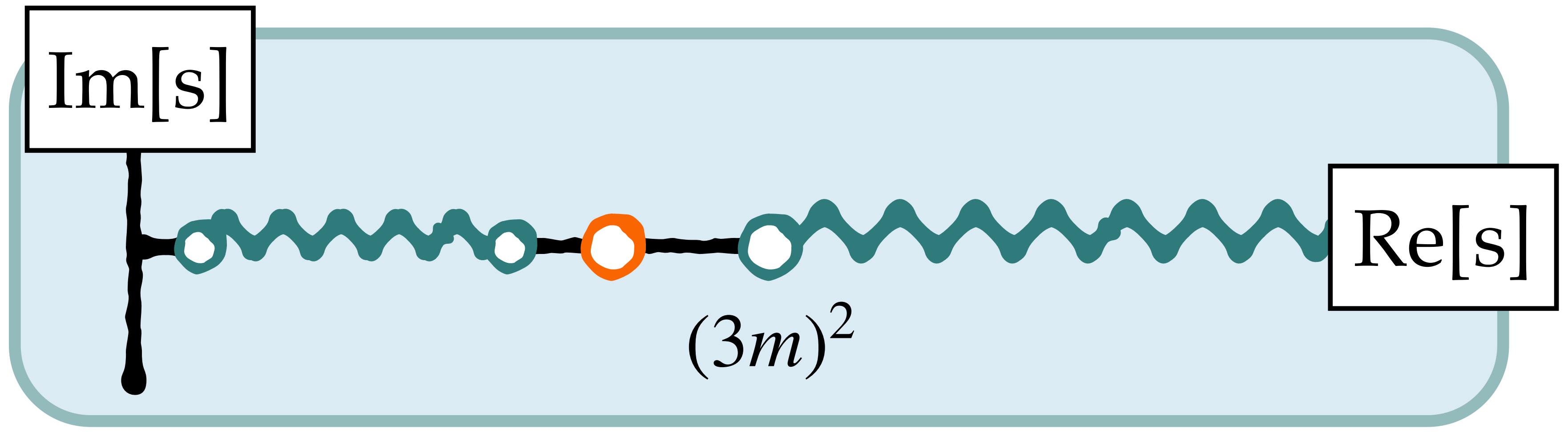


Benefits

- ☑ **analytic description,**
- ☑ **correct singular behavior,**
- ☑ **infinite-volume Minkowski observables**

Limitations

- ☐ **unknown real functions**



EFTs can be understood as a subset of this

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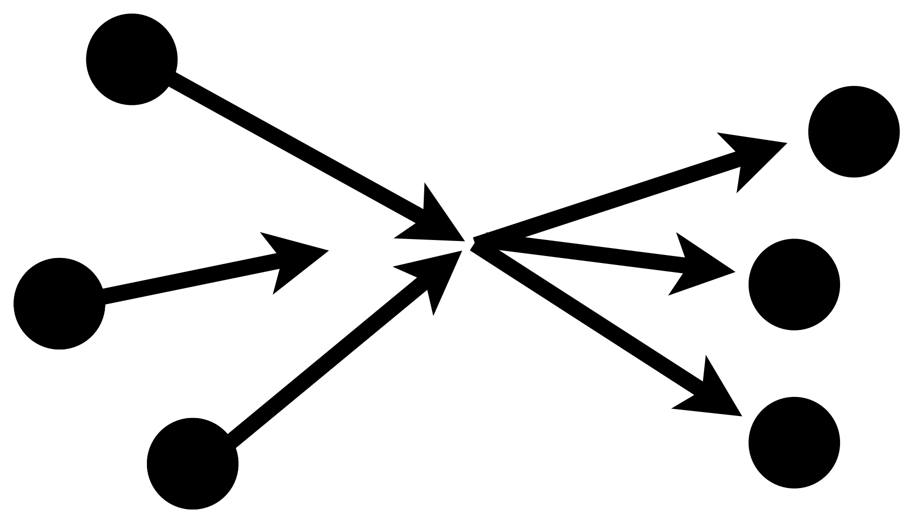

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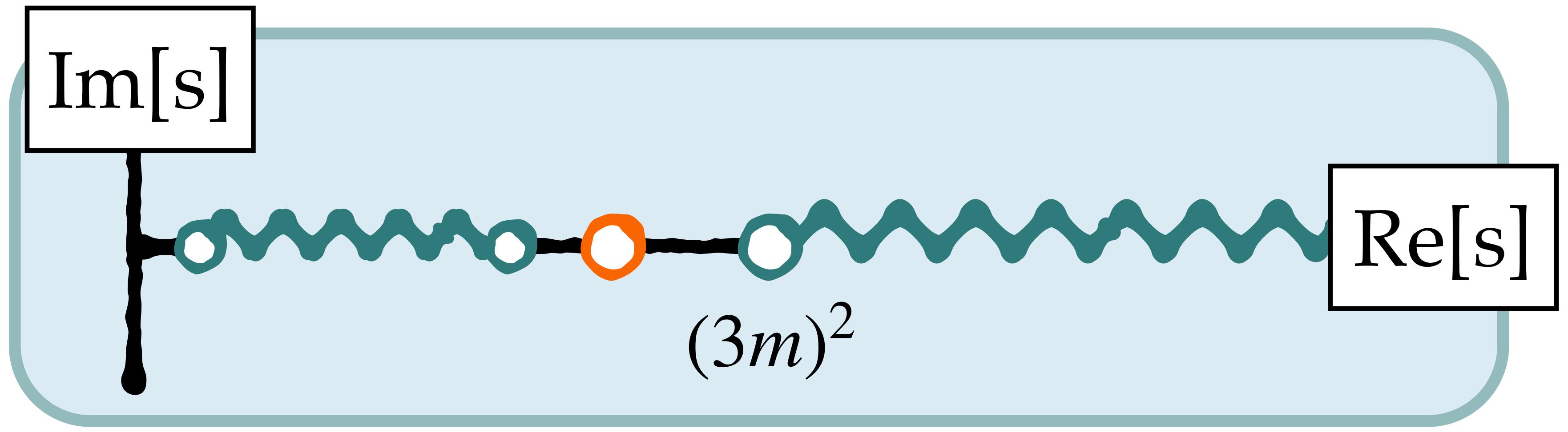
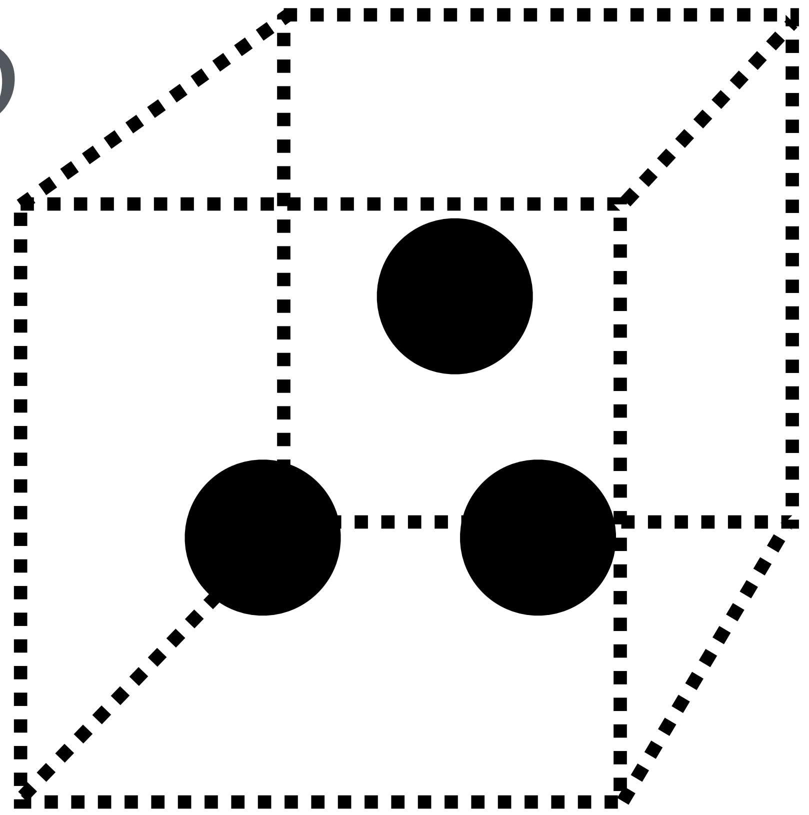


Lattice QCD

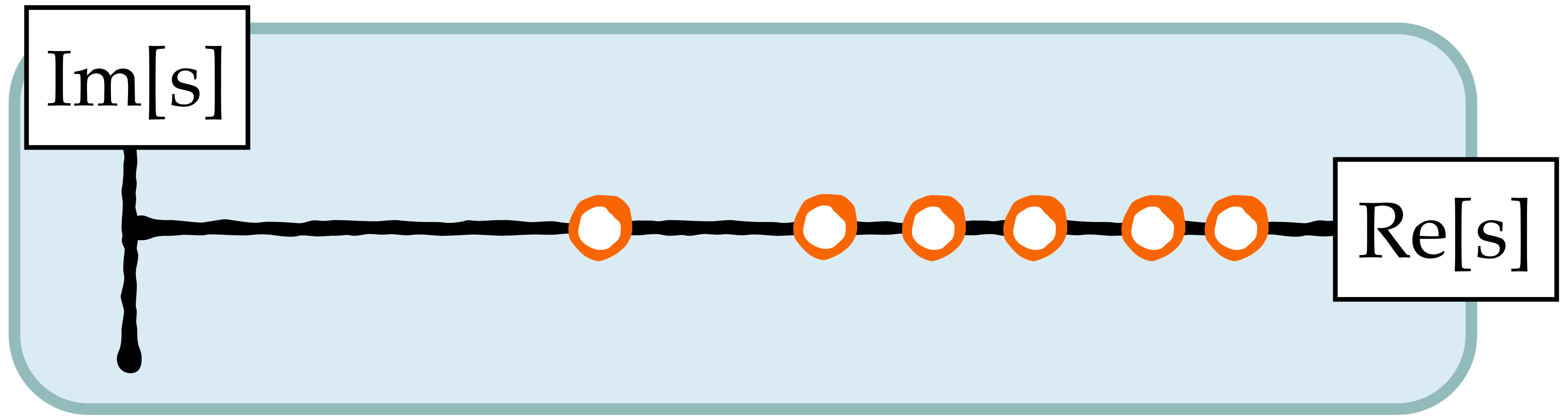


Benefits

- ☑ **treats dynamics exactly,**

Limitations

- ☐ **computationally costly**
- ☐ **finite Euclidean spacetime**
- ☐ **no asymptotic states**



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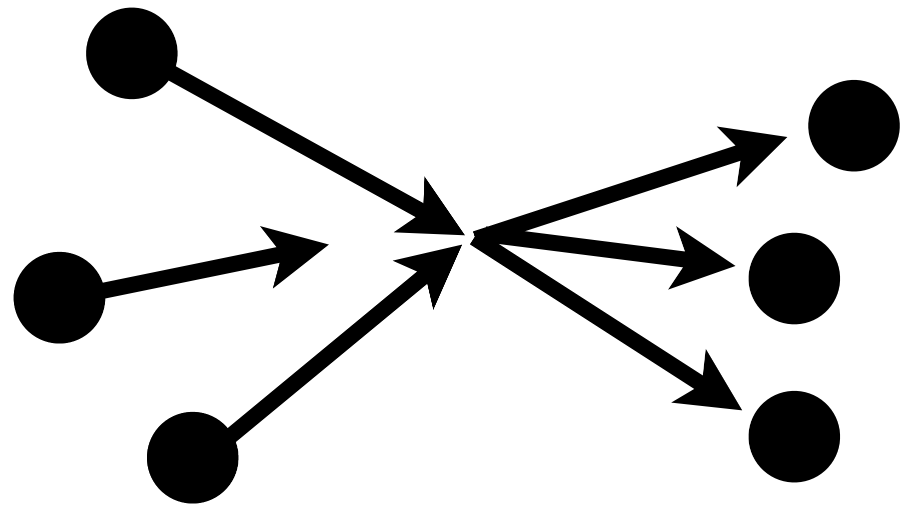
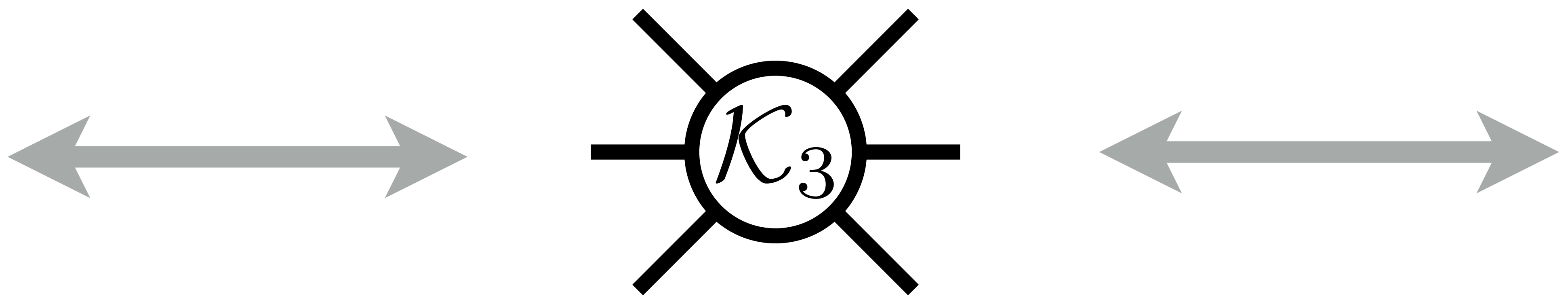
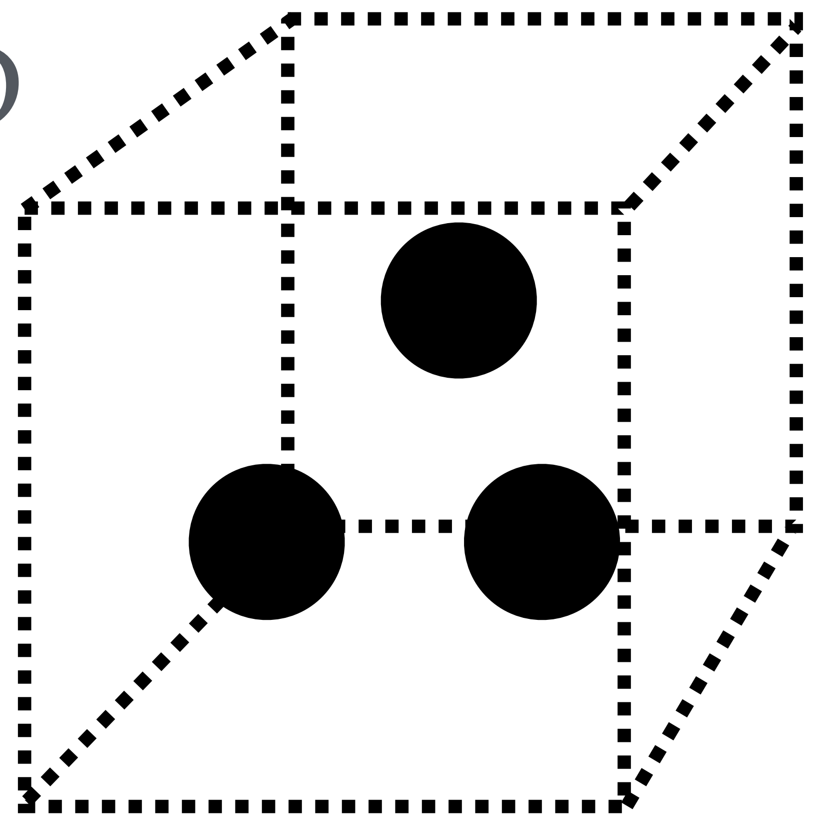




short-distance dynamics

Lattice QCD



nearly a continuum of references:

Rusetsky & Polejaeva(2012)

RB & Davoudi (2012)

Hansen & Sharpe (2014+)

RB, Hansen, Sharpe, …(2017+)

Mai & Doring (2017)

…

Jackura & RB (2023)

RB, Jackura & Costa (to appear)

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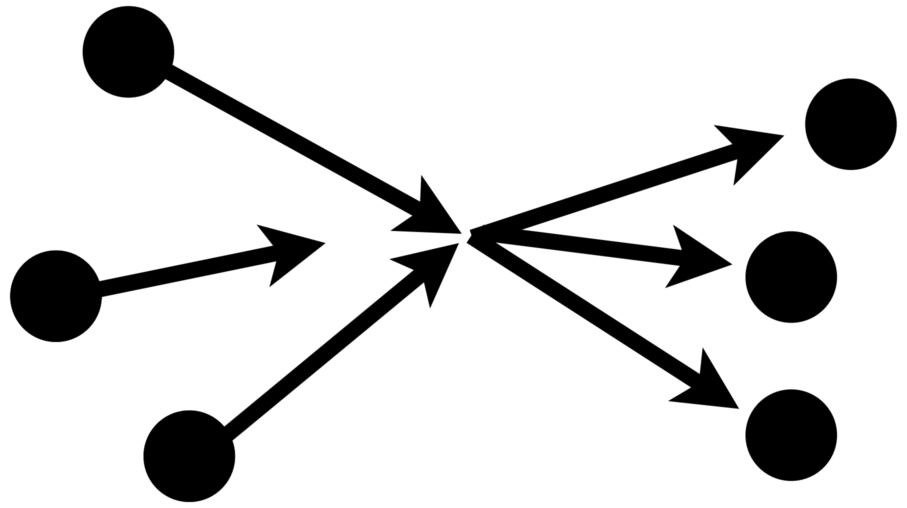

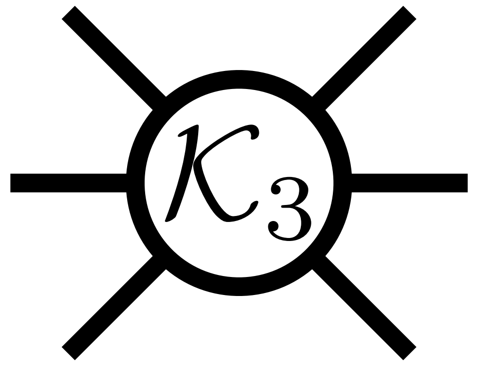

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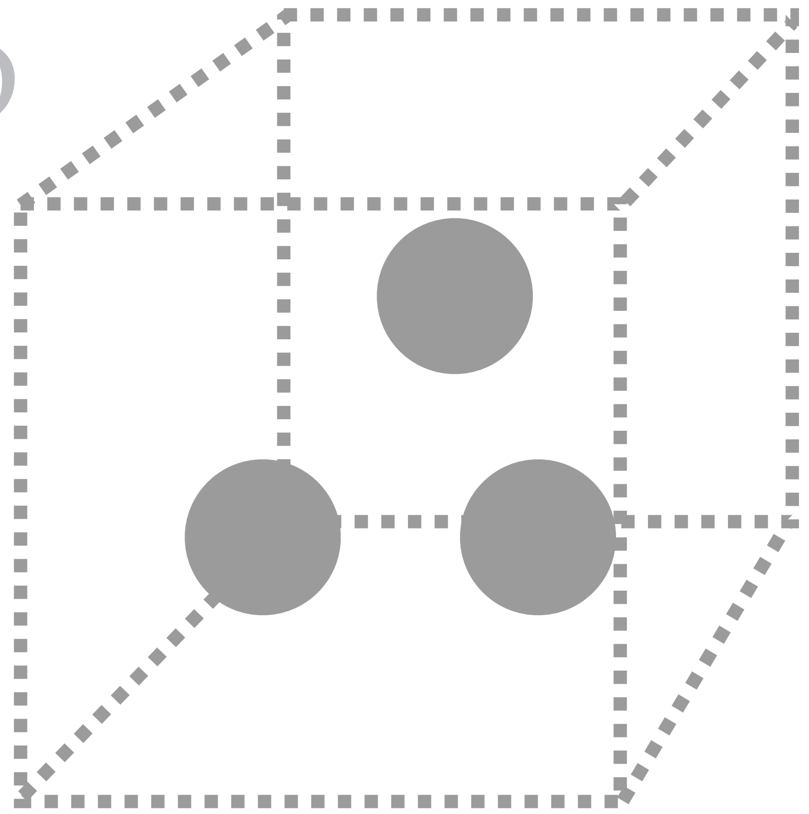
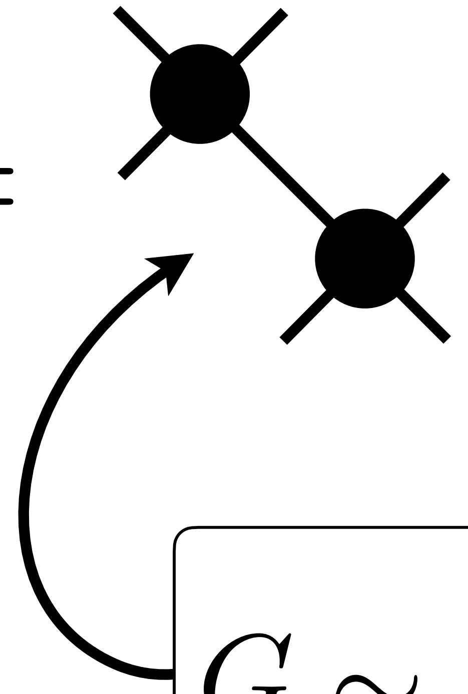


$$i\mathcal{M}_3 =$$



$$+ \cdots$$

$$G \sim \frac{1}{(P-p-k)^2 - m^2}$$

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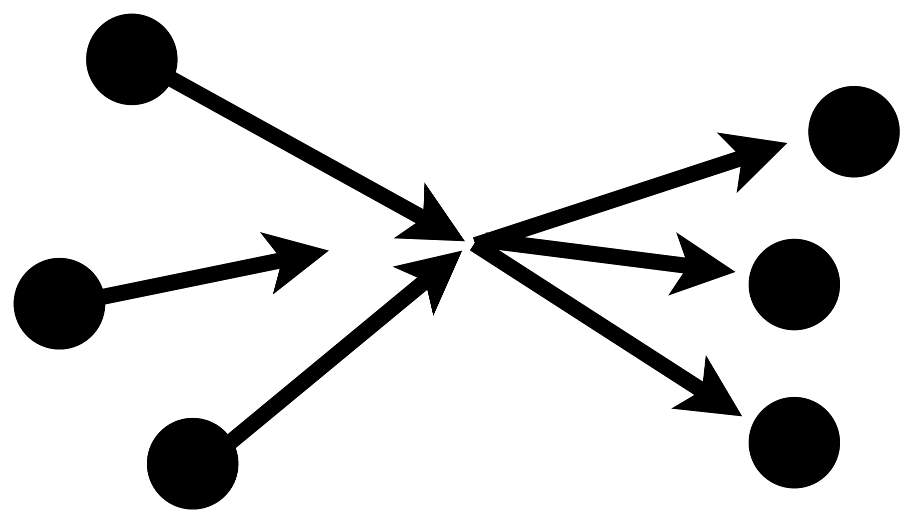

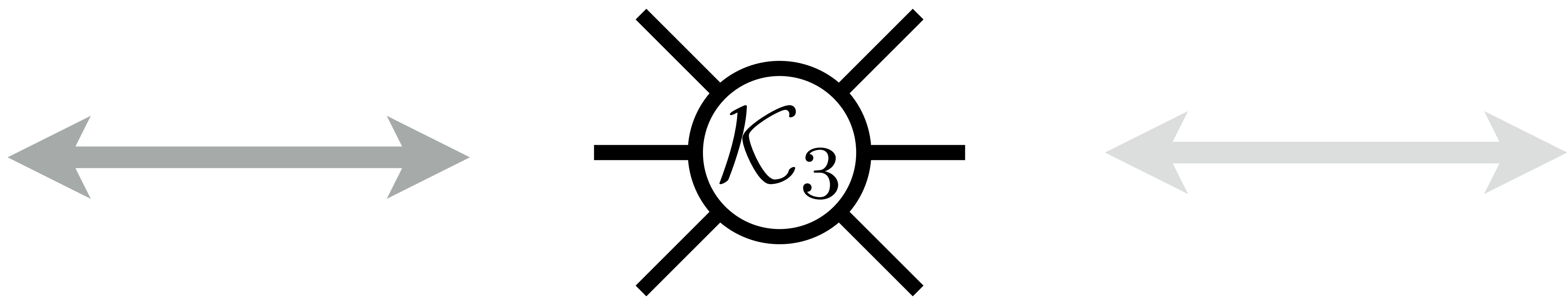

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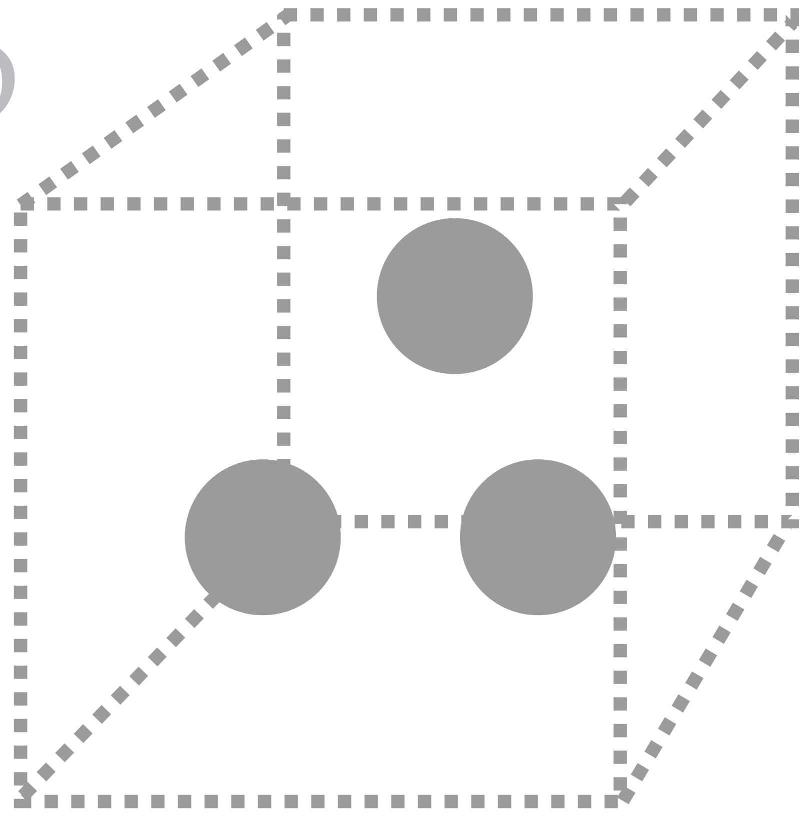
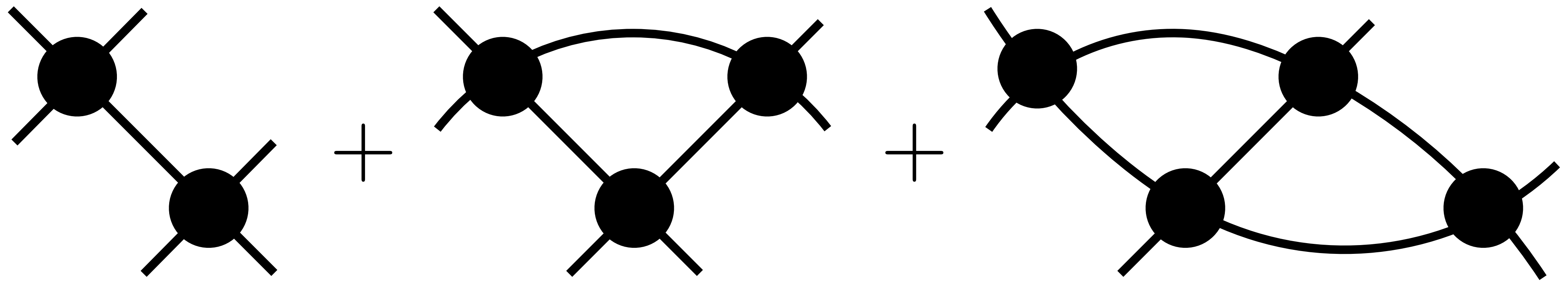


short-distance dynamics

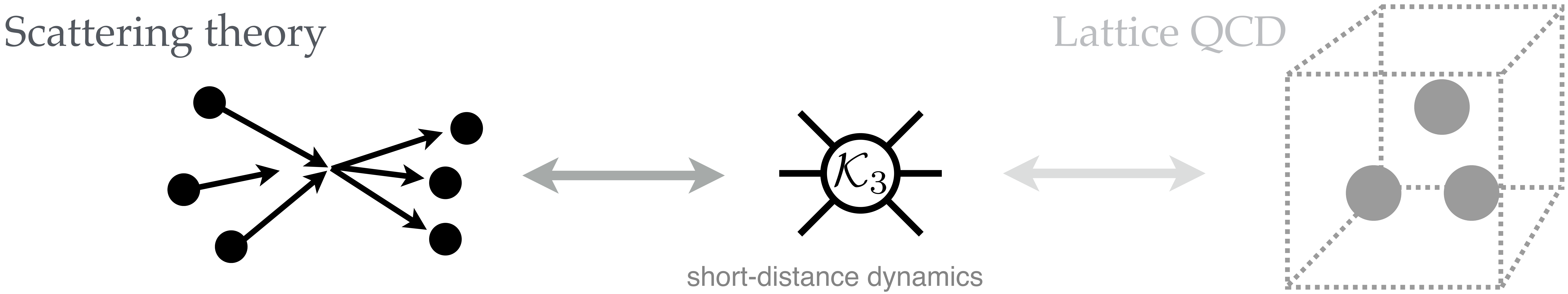
$i\mathcal{M}_3 =$  $+ \cdots$

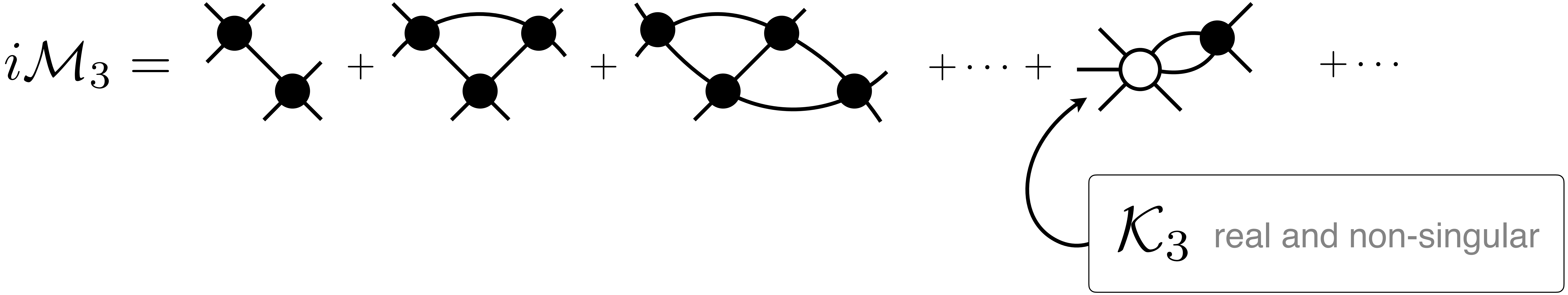
satisfies an integral equation

Where $\mathcal{D} = \mathcal{M}_2 \, d \, \mathcal{M}_2$ and

$$d = -G - \int G \, \mathcal{M}_2 d$$

Arsenal of non-perturbative tools





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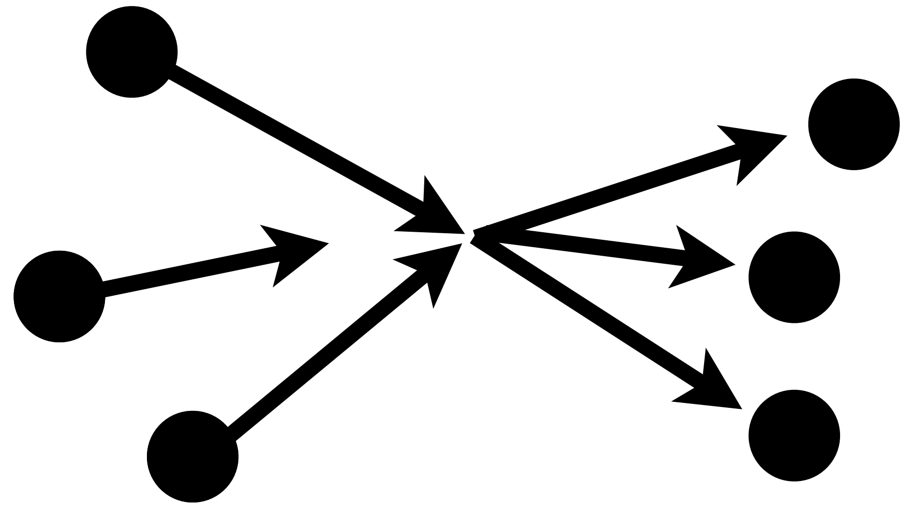

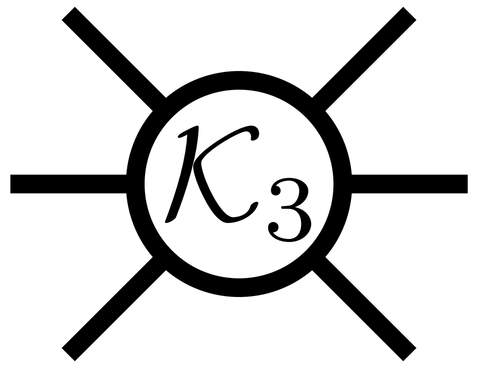

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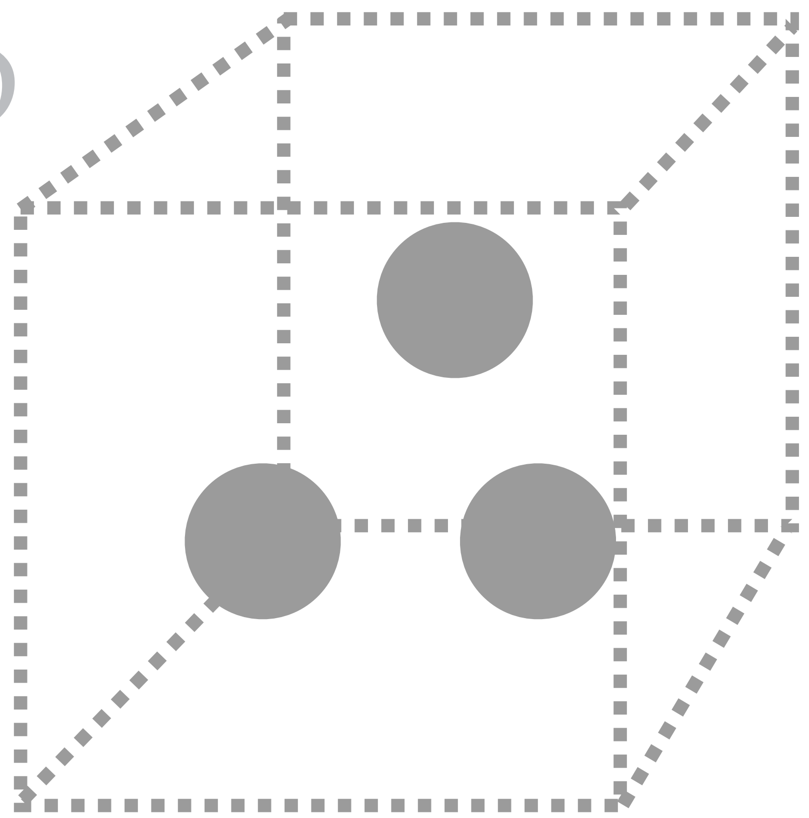
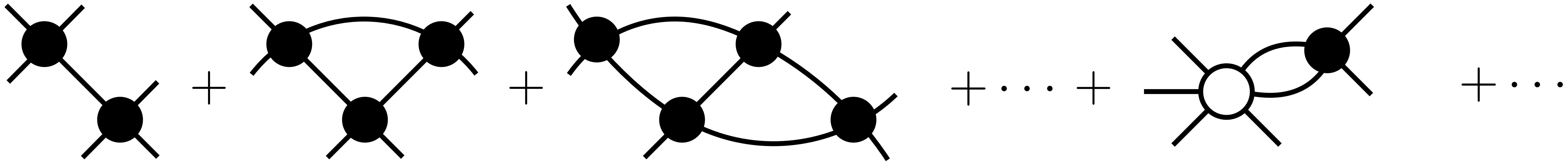


$i\mathcal{M}_3 =$ 

$$= i\mathcal{D} + i\mathcal{L}[\mathcal{D}] \cdot \mathcal{F}[\mathcal{D}, \mathcal{K}_3] \cdot \mathcal{R}[\mathcal{D}]$$

Integral equations

We need to solve:

$$d(\mathbf{p}', s, \mathbf{p}) = -G(\mathbf{p}', s, \mathbf{p}) - \int_0^{q_{\max}} \underbrace{\frac{d^3\mathbf{q}}{(2\pi)^3 2\omega_q} G(\mathbf{p}', s, \mathbf{q}) \mathcal{M}_2(q, s)}_{\text{"integration kernel"}} \, d(\mathbf{q}, s, \mathbf{p})$$

Need to resort to numerical solutions.

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Need to resort to numerical solutions.

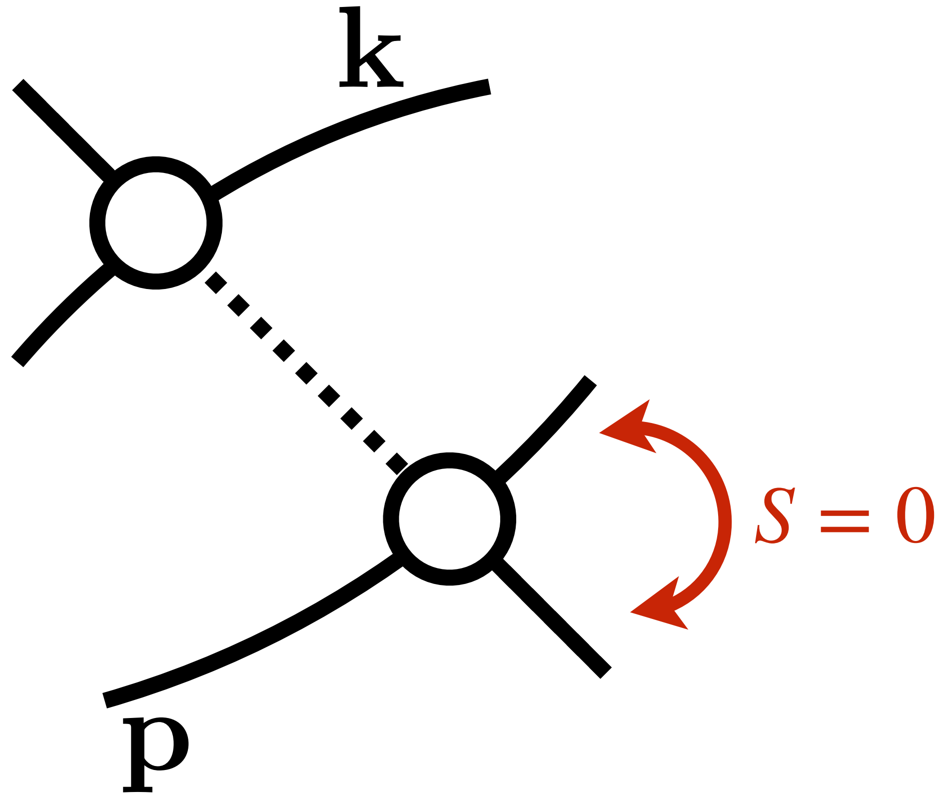
Three correlated challenges:

- ☐ 3D integral equation,
- ☐ need to project to **angular momentum and parity**,
- ☐ integration kernel is generally singular.

Partial wave projections

The one-particle exchange is one of the main sources of singularities.

Let us consider the case where $S = 0$:



$$\sim G(\boldsymbol{p}, \boldsymbol{k}) = \frac{1}{(E - \omega_k - \omega_p) - (\boldsymbol{p} + \boldsymbol{k})^2 - m^2 + i\epsilon}$$

$$= \frac{1}{(E - \omega_k - \omega_p) - k^2 - p^2 - m^2 - 2pk\cos\theta + i\epsilon}$$

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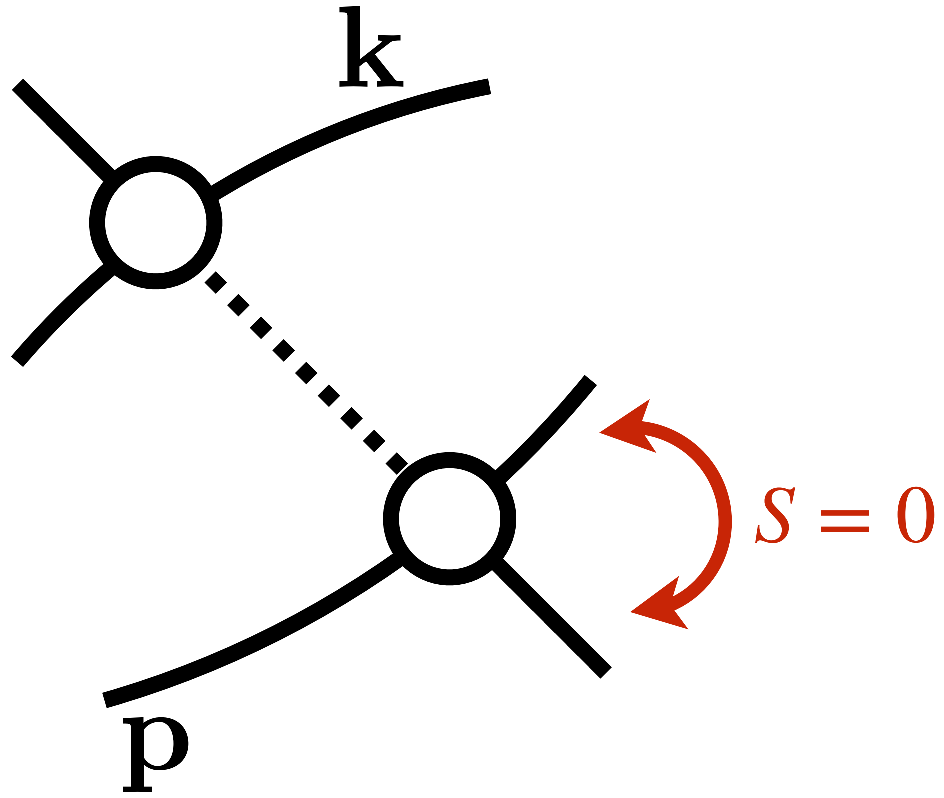

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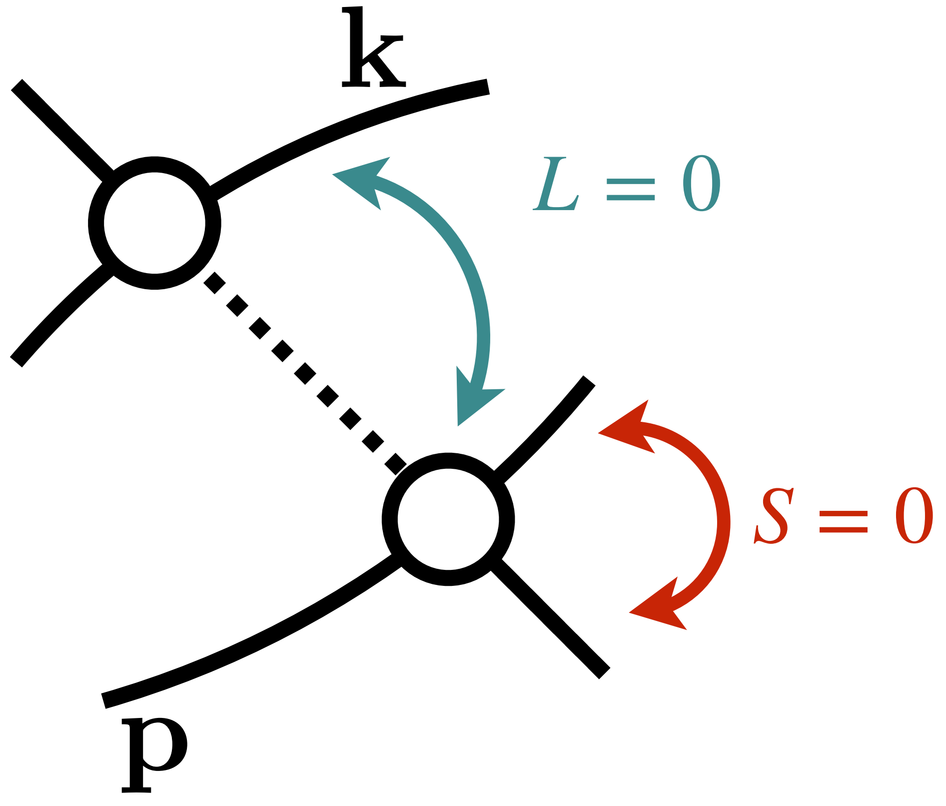
$$= \frac{1}{(E - \omega_k - \omega_p) - k^2 - p^2 - m^2 - 2pk\cos\theta + i\epsilon}$$

Projecting to total $J = 0$ amounts to integrating over all angles:

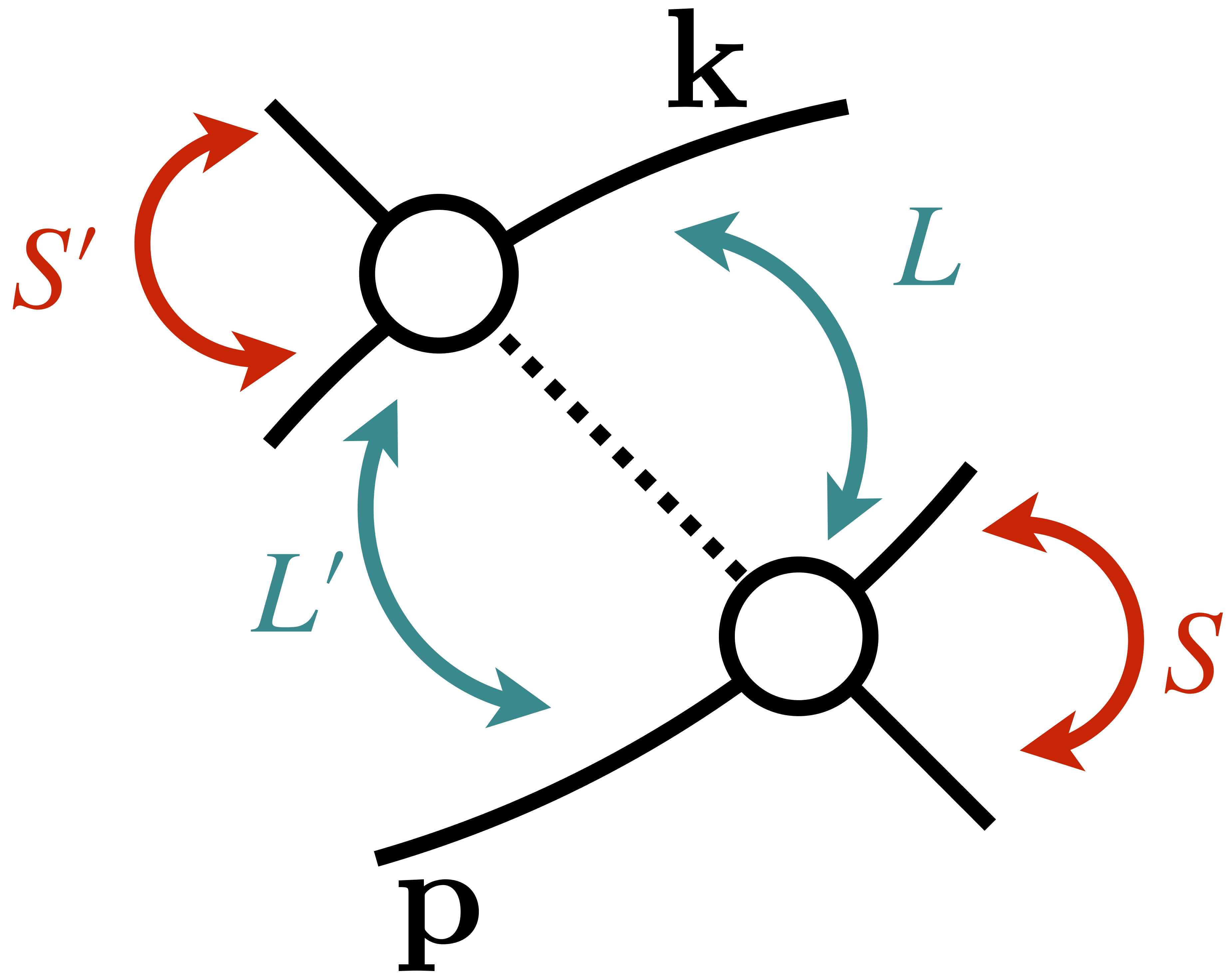


$$\sim G(p, k) = \frac{1}{2}\int_{-1}^{1} d\cos\theta \, G(\boldsymbol{p}, \boldsymbol{k}) = -\frac{1}{4pk}\log\frac{z_{pk} - 1}{z_{pk} + 1}$$

$$z(p, k) = \frac{(E - \omega_k - \omega_p)^2 - k^2 - p^2 - m^2}{2pk}$$

Partial wave projections

In general…



$$\left[\mathcal{G}^{J^P}\right]_{L'S',LS} = \left[\mathcal{K}_{\mathcal{G}}^{J^P}\right]_{L'S',LS} + \left[\mathcal{T}^{J^P}\right]_{L'S',LS} Q_0(\zeta_{pk})$$

known kinematic functions

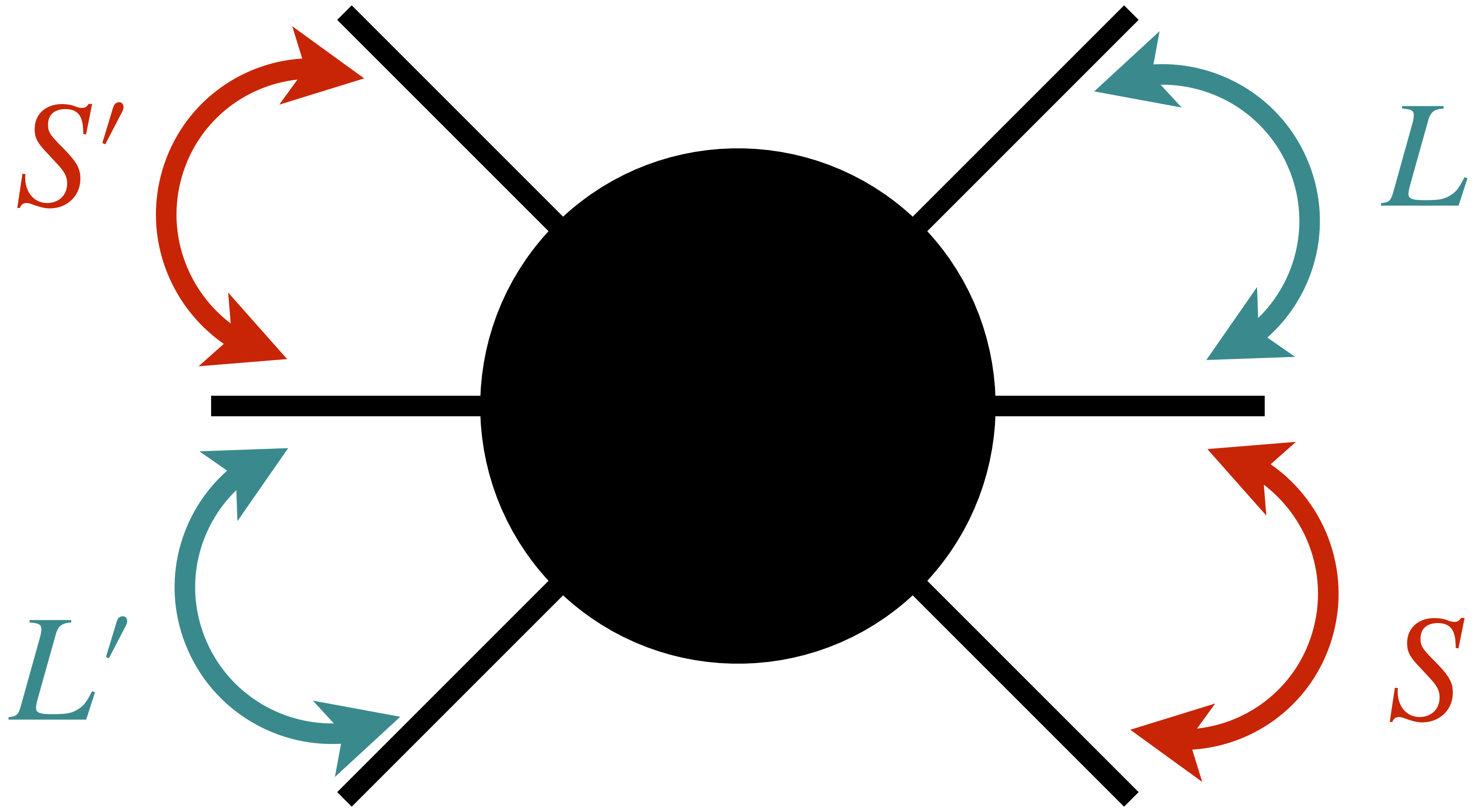
Legendre functions

Jackura, RB (2023)

$$Q_0(\zeta) = \frac{1}{2}\log\left(\frac{\zeta+1}{\zeta-1}\right)$$

Partial wave projections

In general…



$$= i \left[\mathcal{M}_3^{J^P} \right]_{L'S',LS}$$

S. R. Costa

Jackura

Costa, Jackura, RB (2024)

outline

- [x] integral equations
- [x] angular momentum projection
- [ ] finite-volume formalism
- [ ] a lattice QCD calculation
- [ ] toy model calculations
- [ ] Efimov physics
- [ ] consistency checks and the breakdown of Lüscher

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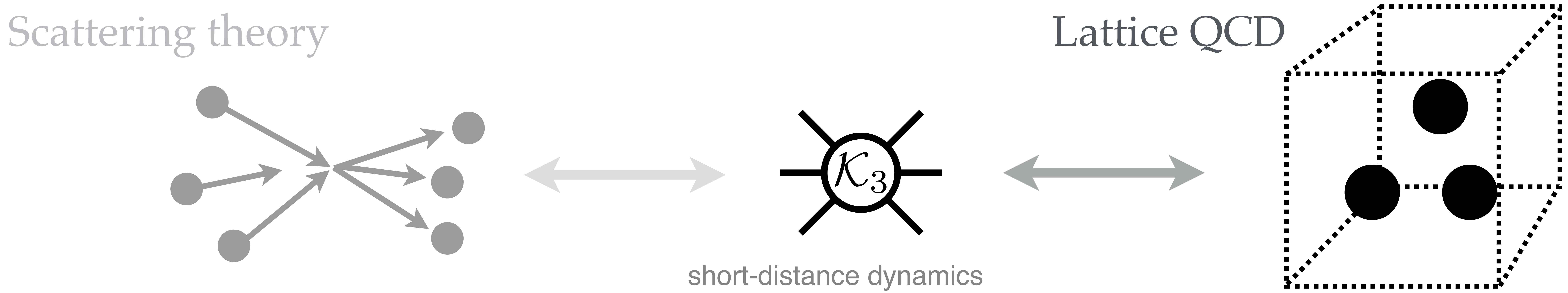
Arsenal of non-perturbative tools



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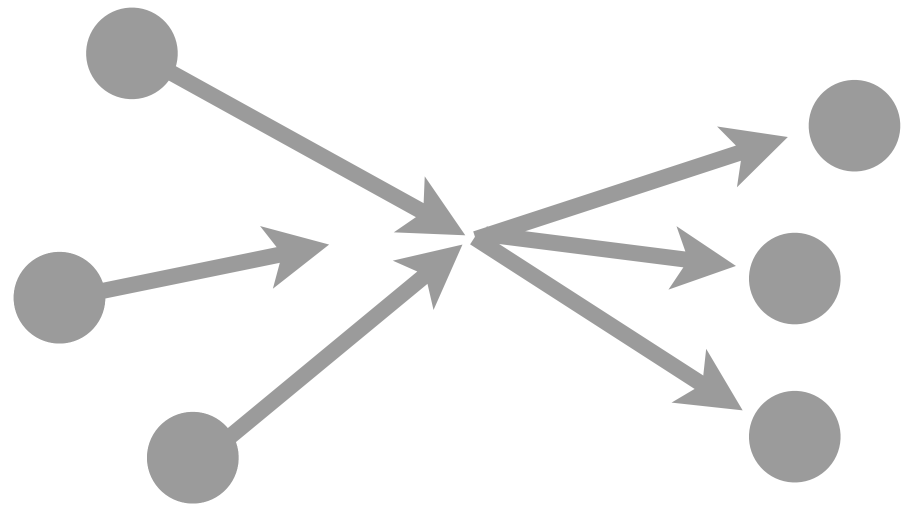
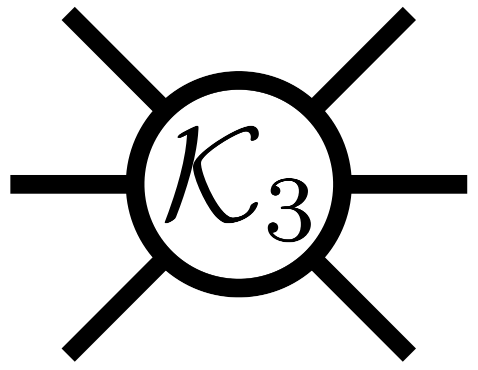
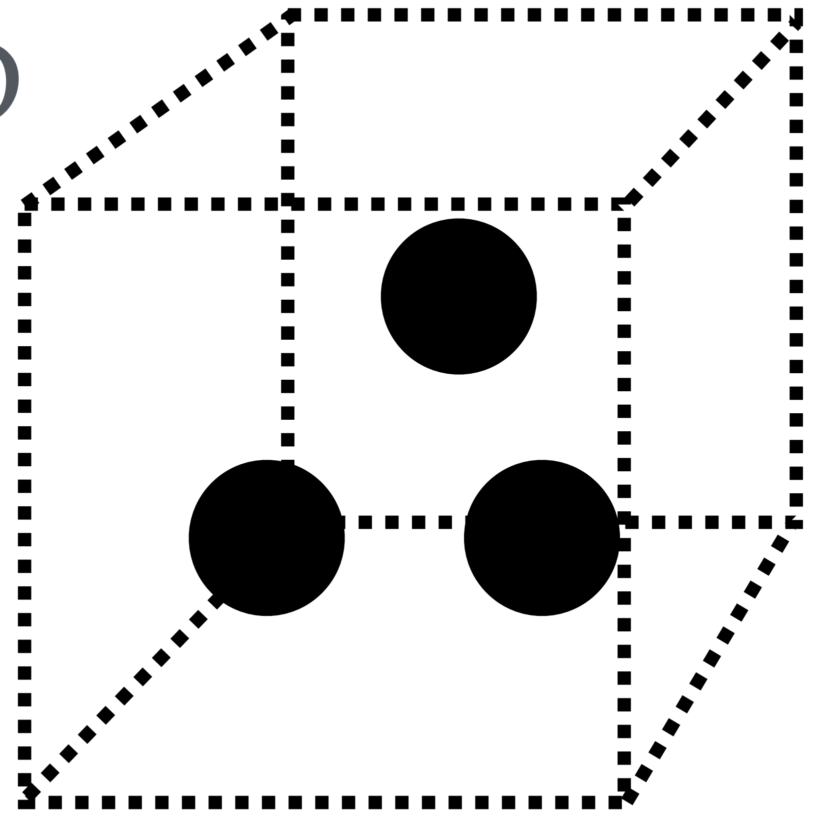
Scattering theory





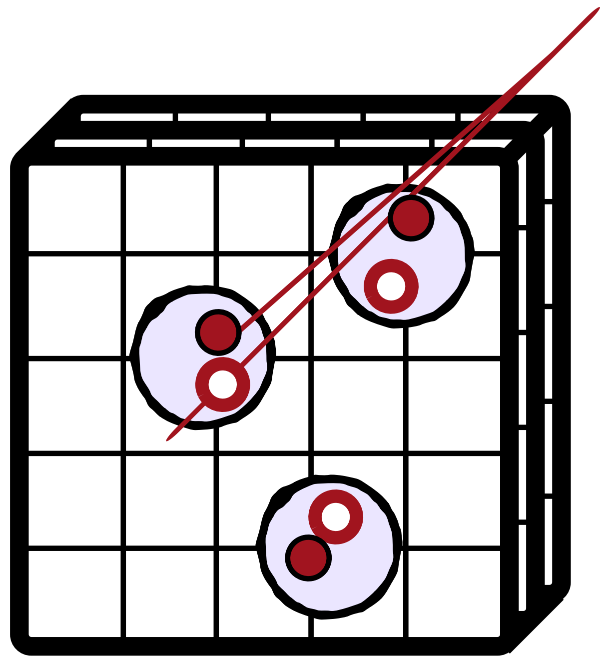
short-distance dynamics

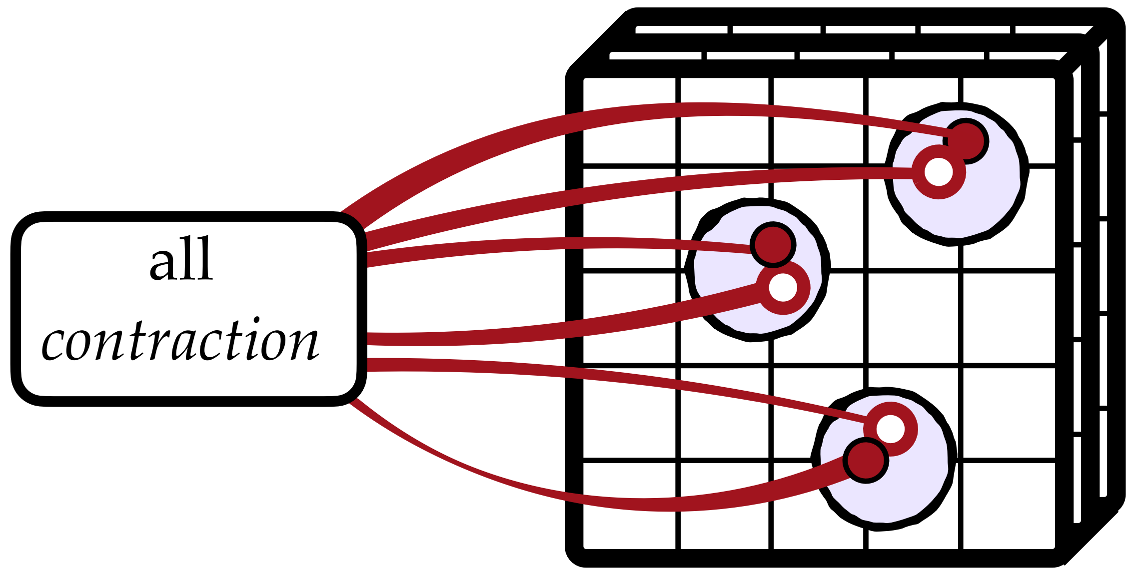
Lattice QCD



☑ Two point correlation functions:

$$C(t) = \langle 0 | \mathcal{O}(t)\mathcal{O}^\dagger(0) | 0 \rangle = \sum_n c_n\, e^{-E_n t} =$$





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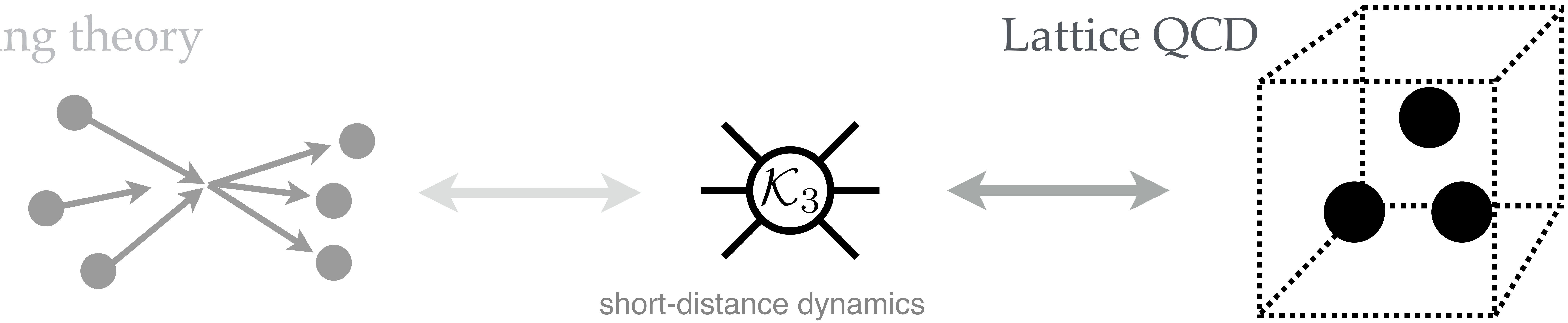

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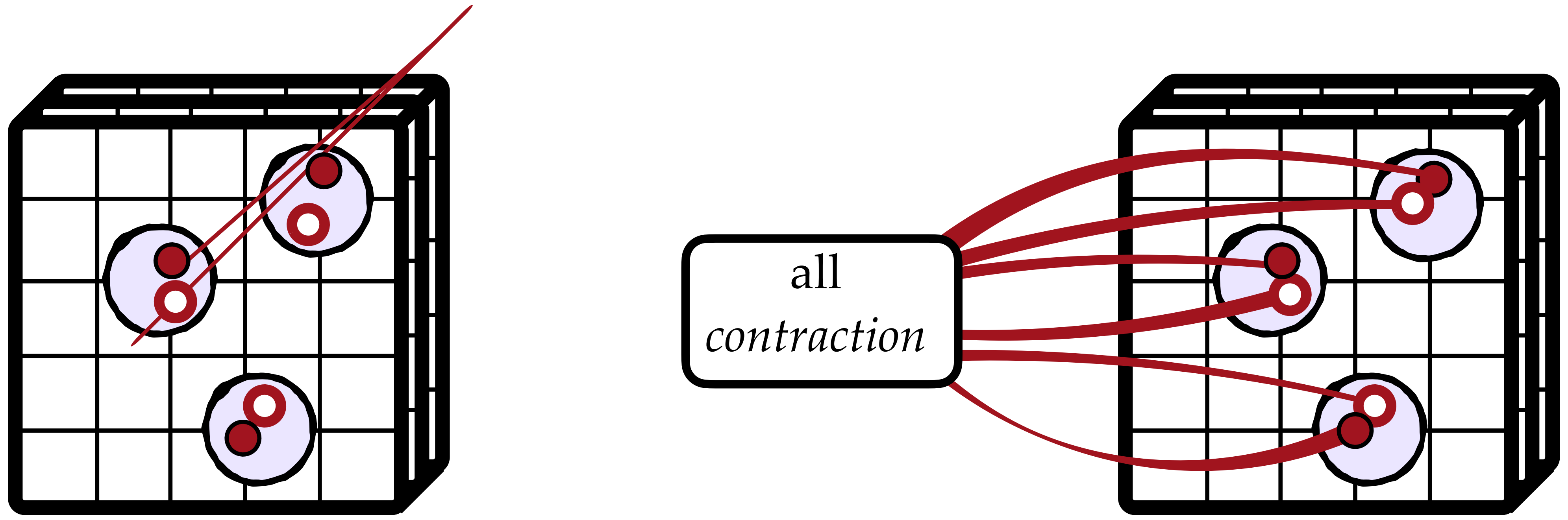


☑ The energy of three *identical spinless bosons* in a box satisfies:

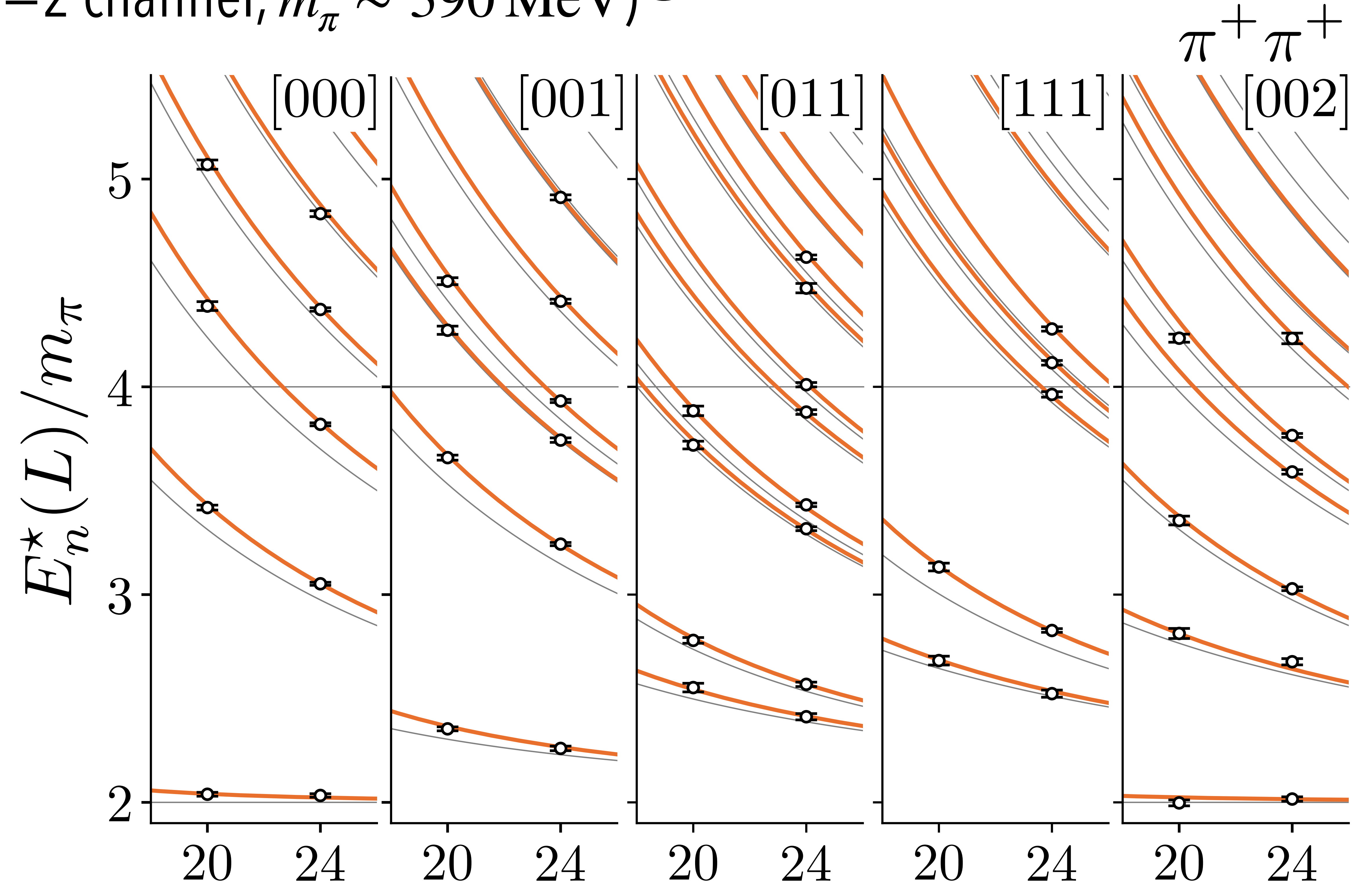
$$F_3^{-1}(P_n, L) + \mathcal{K}_3(P_n^2) = 0 + \mathcal{O}\left(e^{-mL}\right)$$

[up to details I won't go into 🤓]

Hansen & Sharpe (2014+)

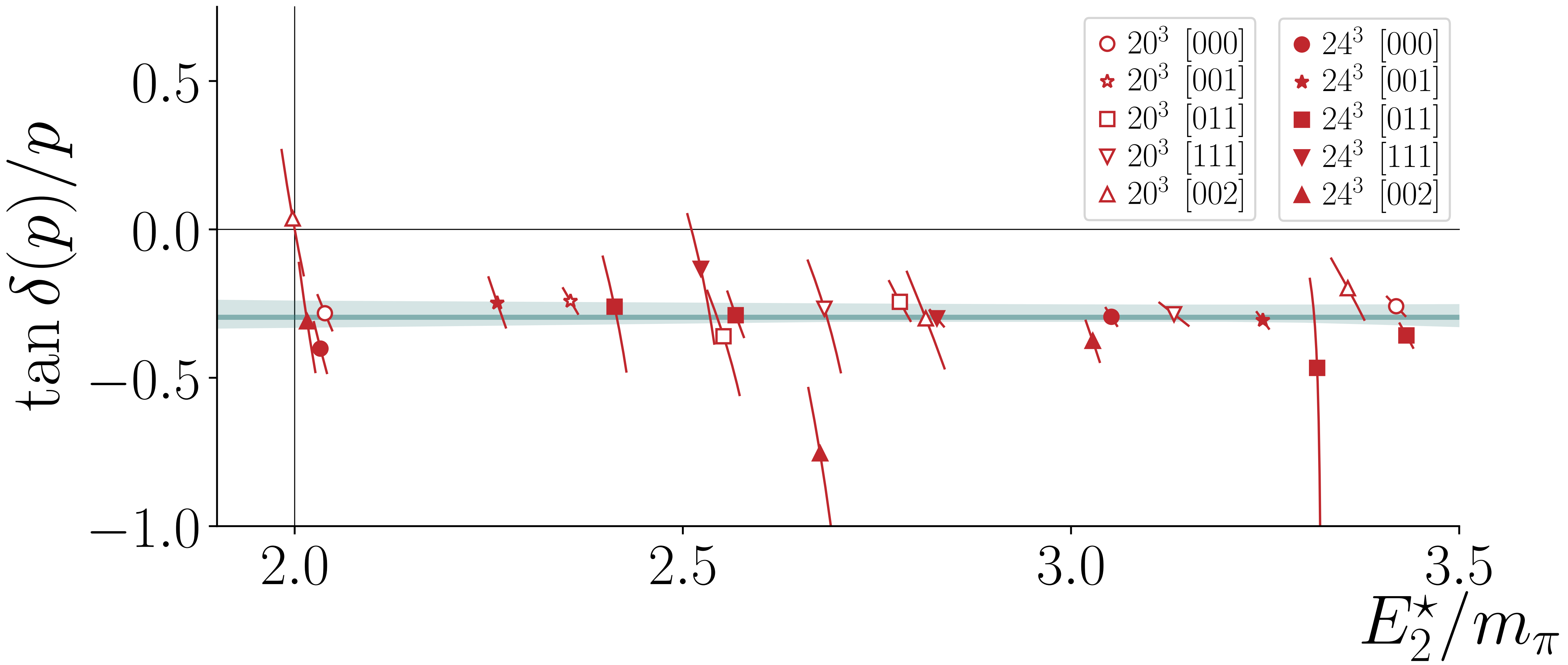
$\pi\pi$ scattering

(I=2 channel, $m_\pi \sim 390\,\mathrm{MeV}$)



$$\det[F^{-1}(P,L) + \mathcal{M}(P)] = 0$$

$$\mathcal{M} \sim \frac{1}{p\cot\delta - ip}$$



hadspec

Hansen, RB, Edwards, Thomas, & Wilson (2020)

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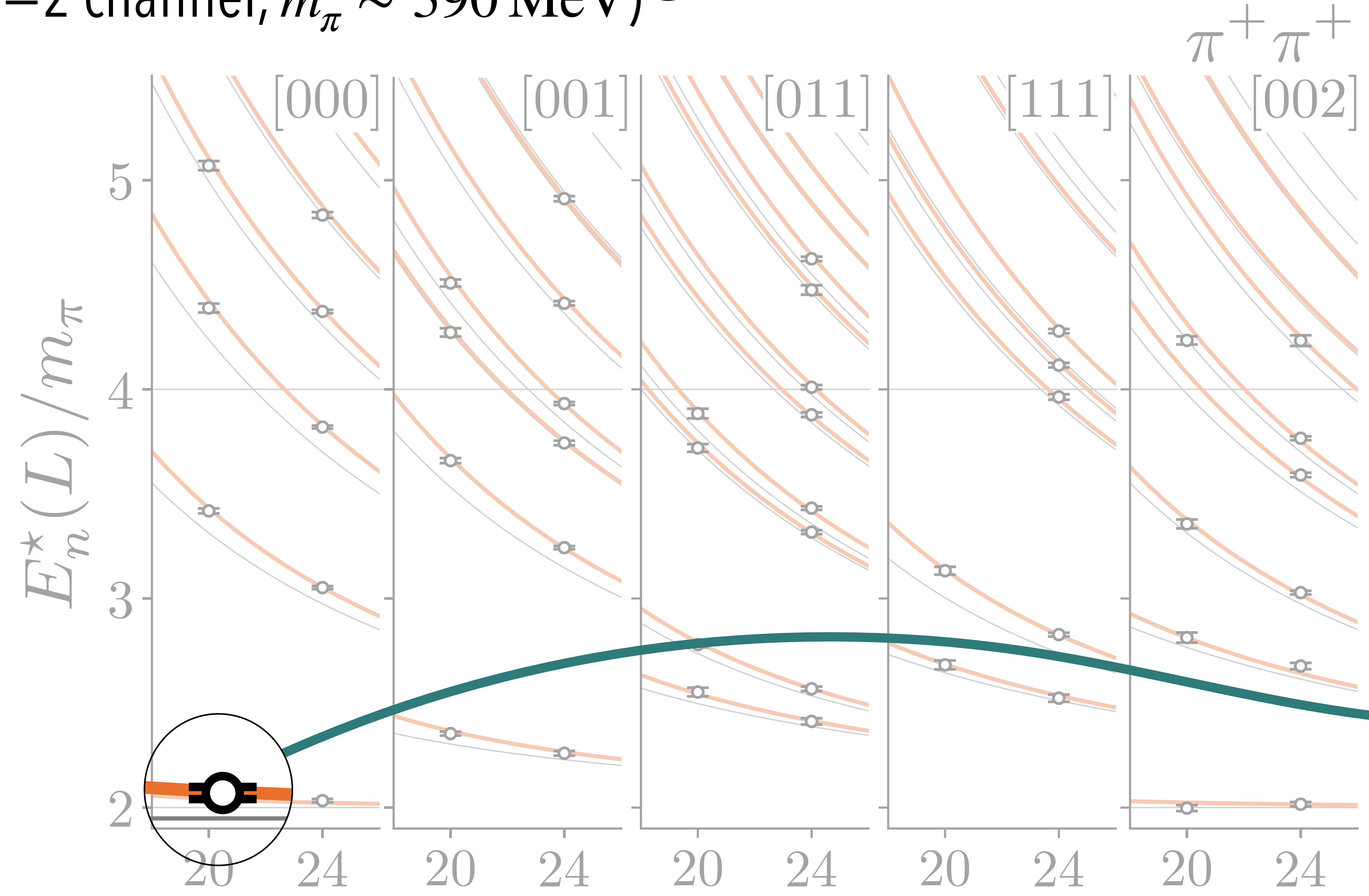

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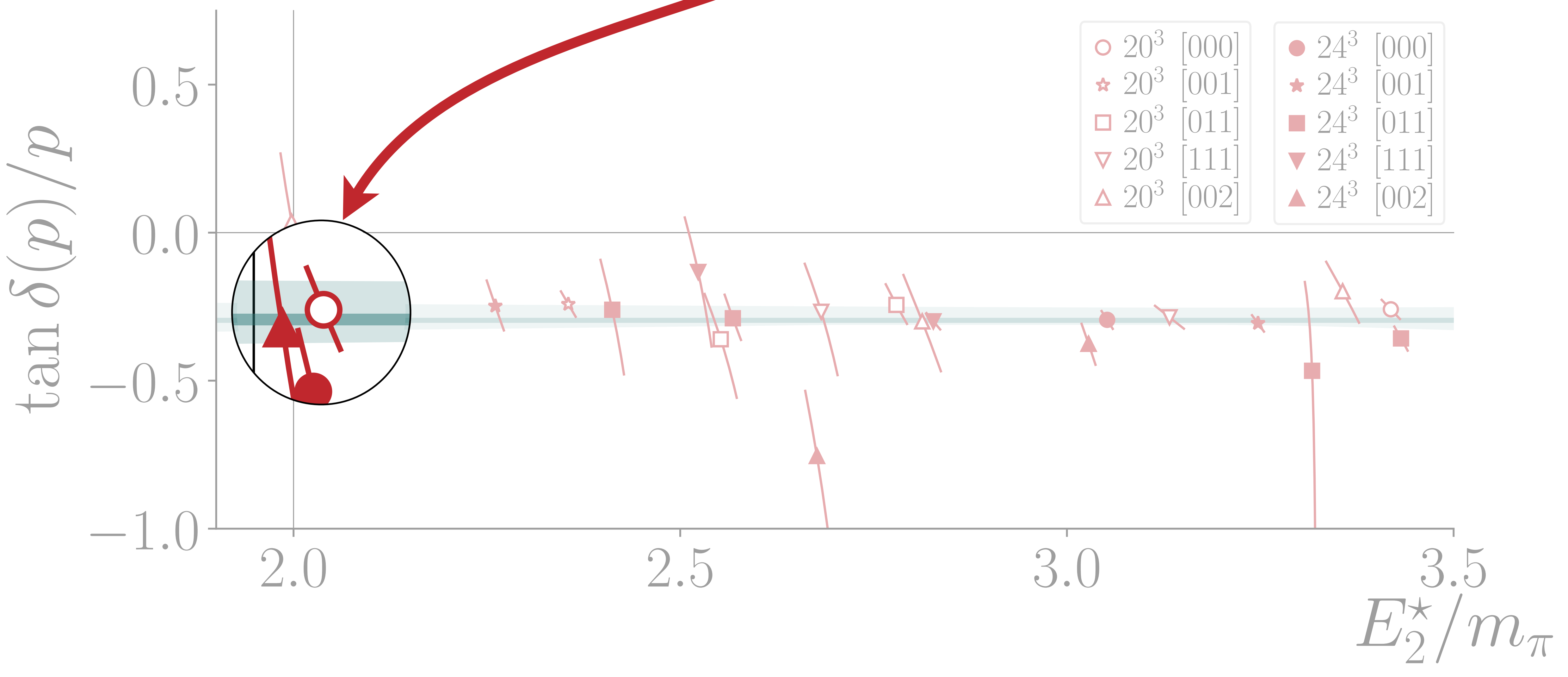
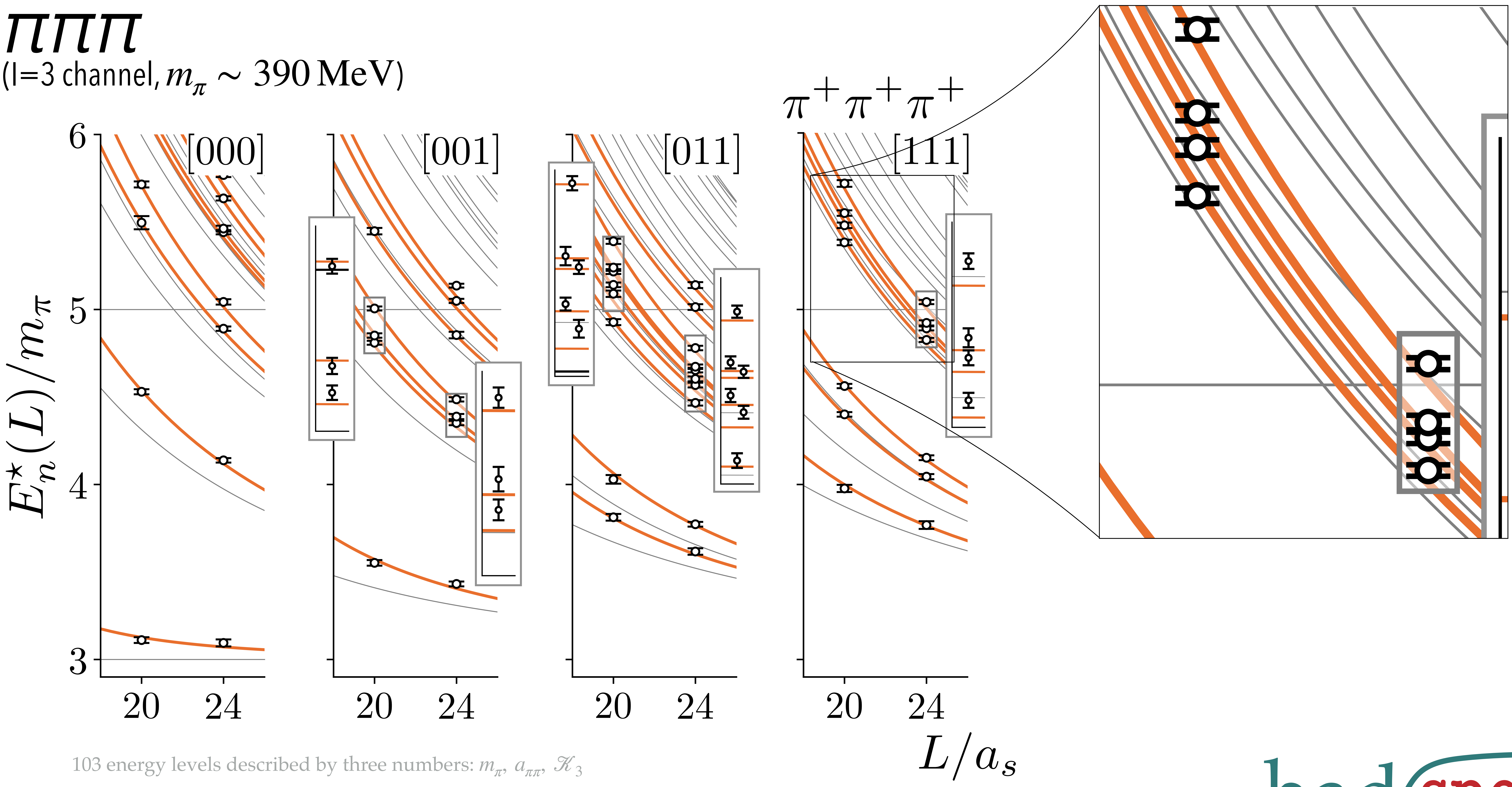


Hansen, RB, Edwards, Thomas, & Wilson (2020)

$\pi\pi\pi$

(I=3 channel, $m_\pi \sim 390\,\mathrm{MeV}$)



103 energy levels described by three numbers: m_π, $a_{\pi\pi}$, $\mathscr{K}_3$

Hansen, RB, Edwards, Thomas, & Wilson (2020)

hadspec

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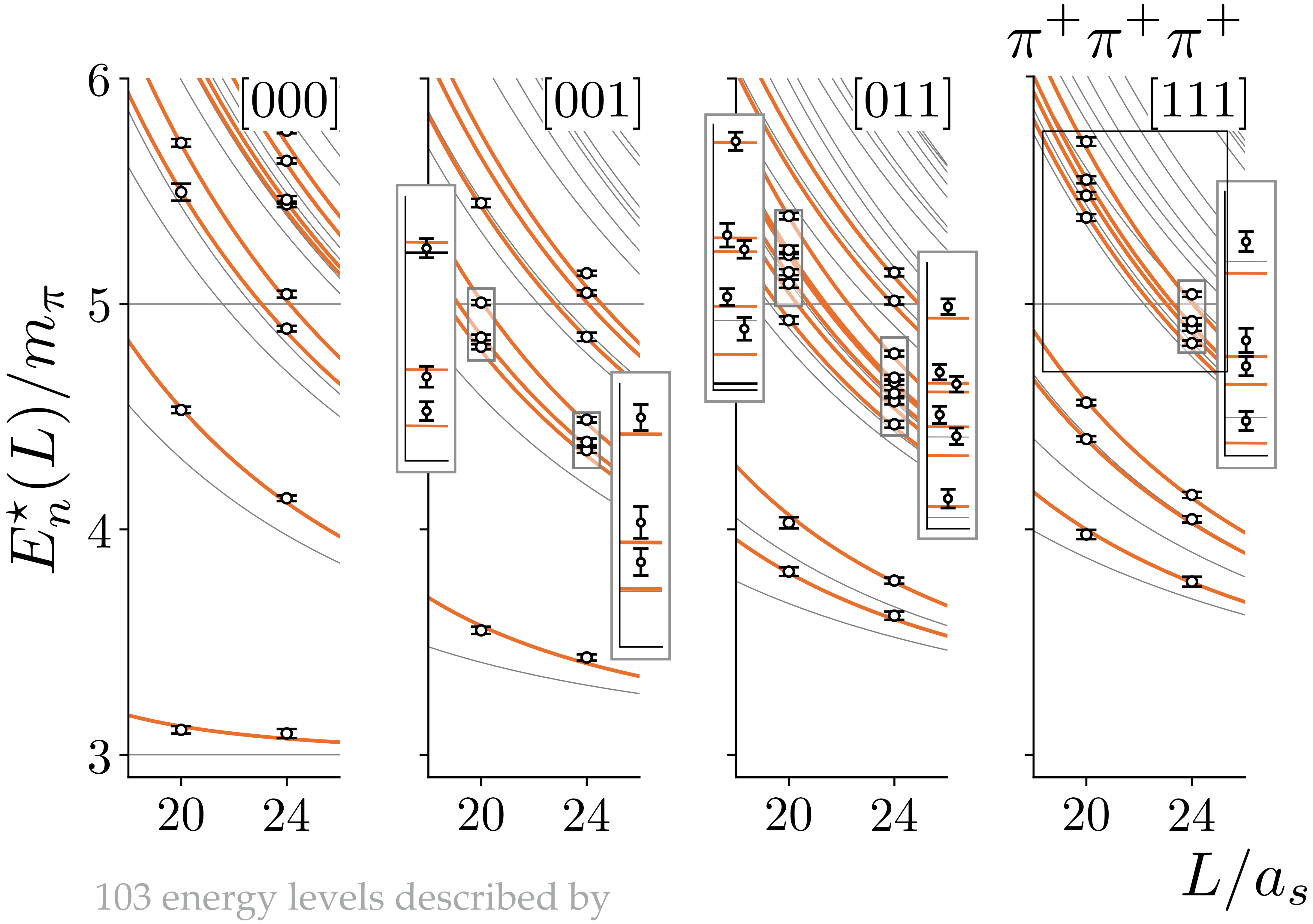
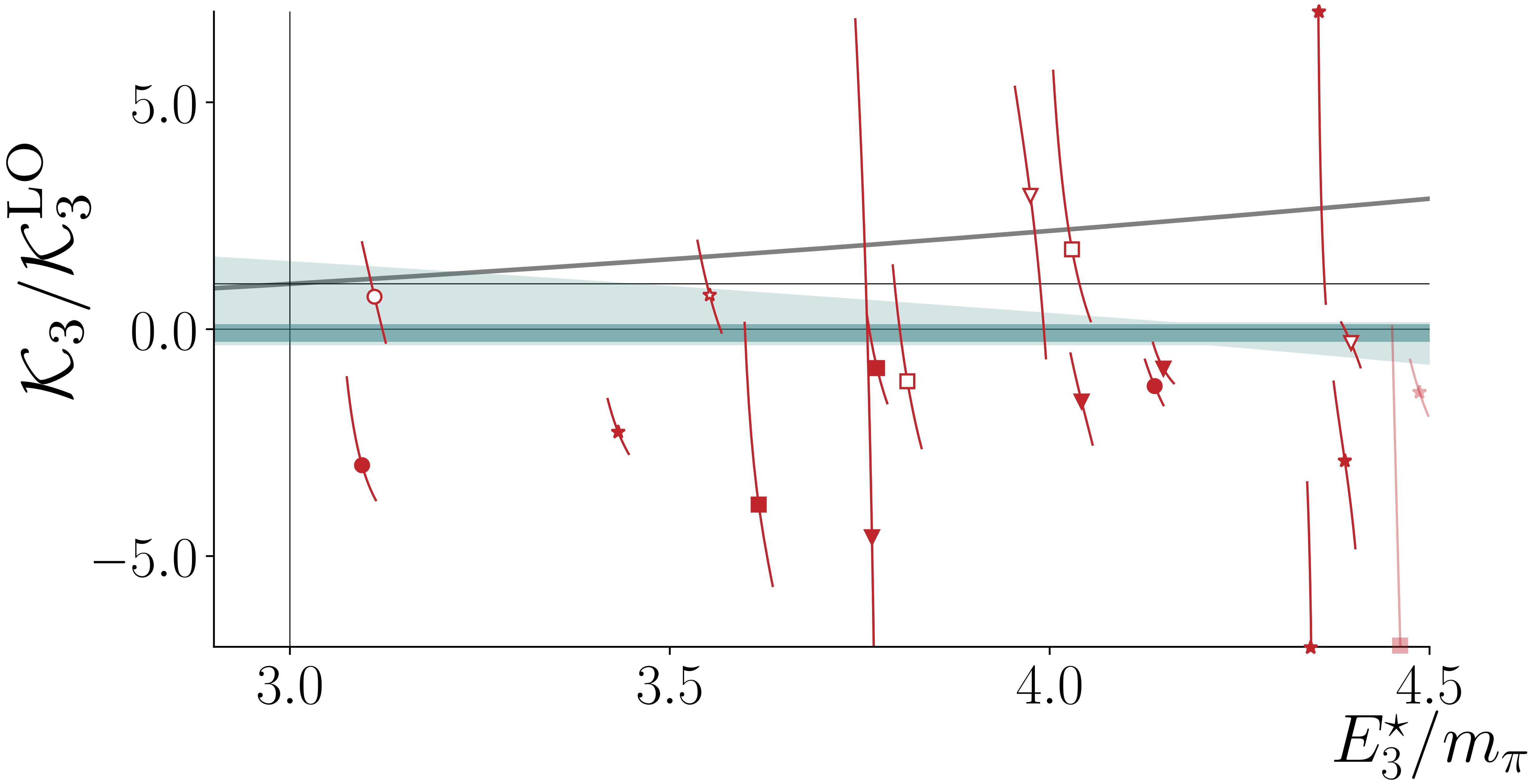


103 energy levels described by three numbers: m_π, $a_{\pi\pi}$, $\mathcal{K}_3$

$$F_3^{-1}(P,L) + \mathcal{K}_3(P^2) = 0$$



Hansen, RB, Edwards, Thomas, & Wilson (2020)

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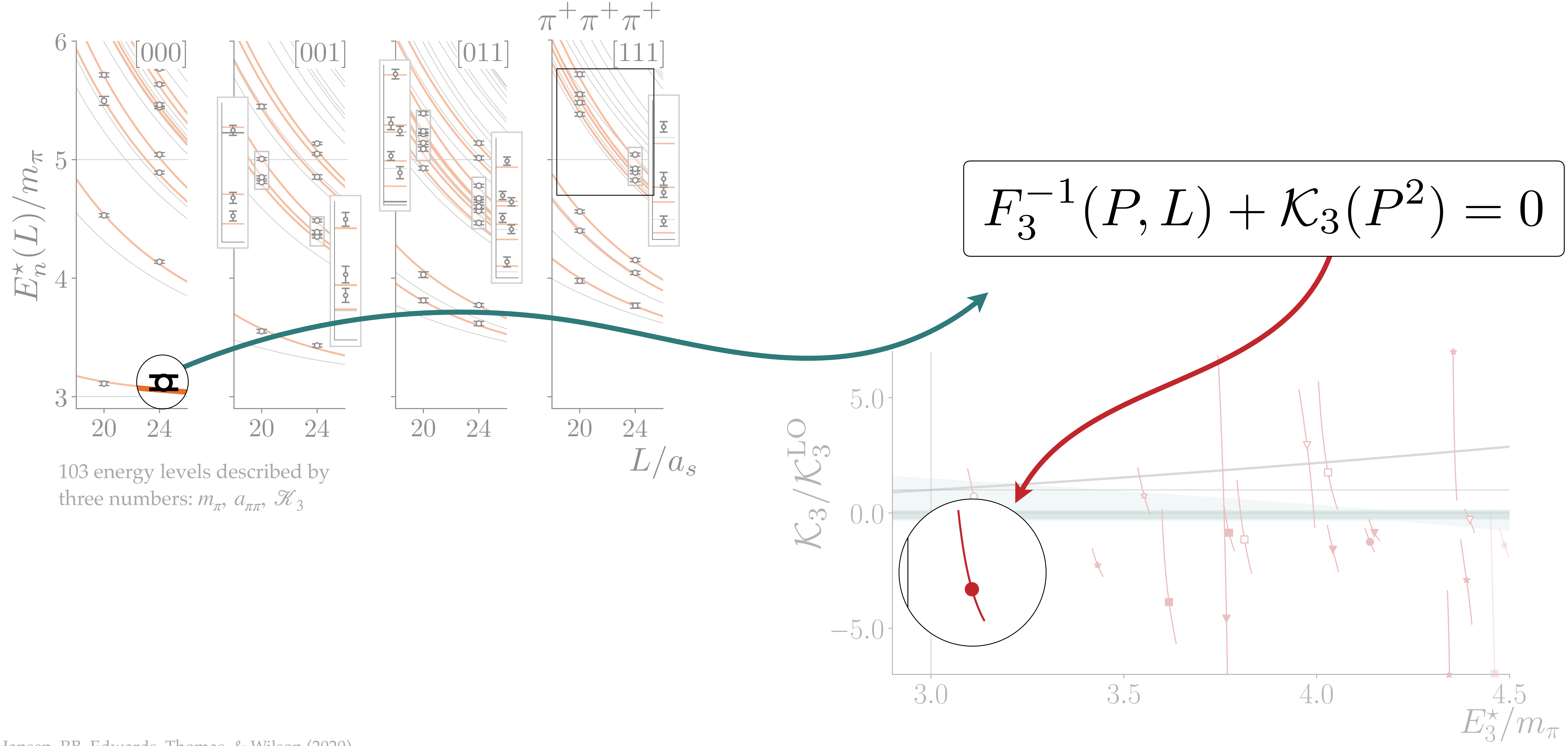

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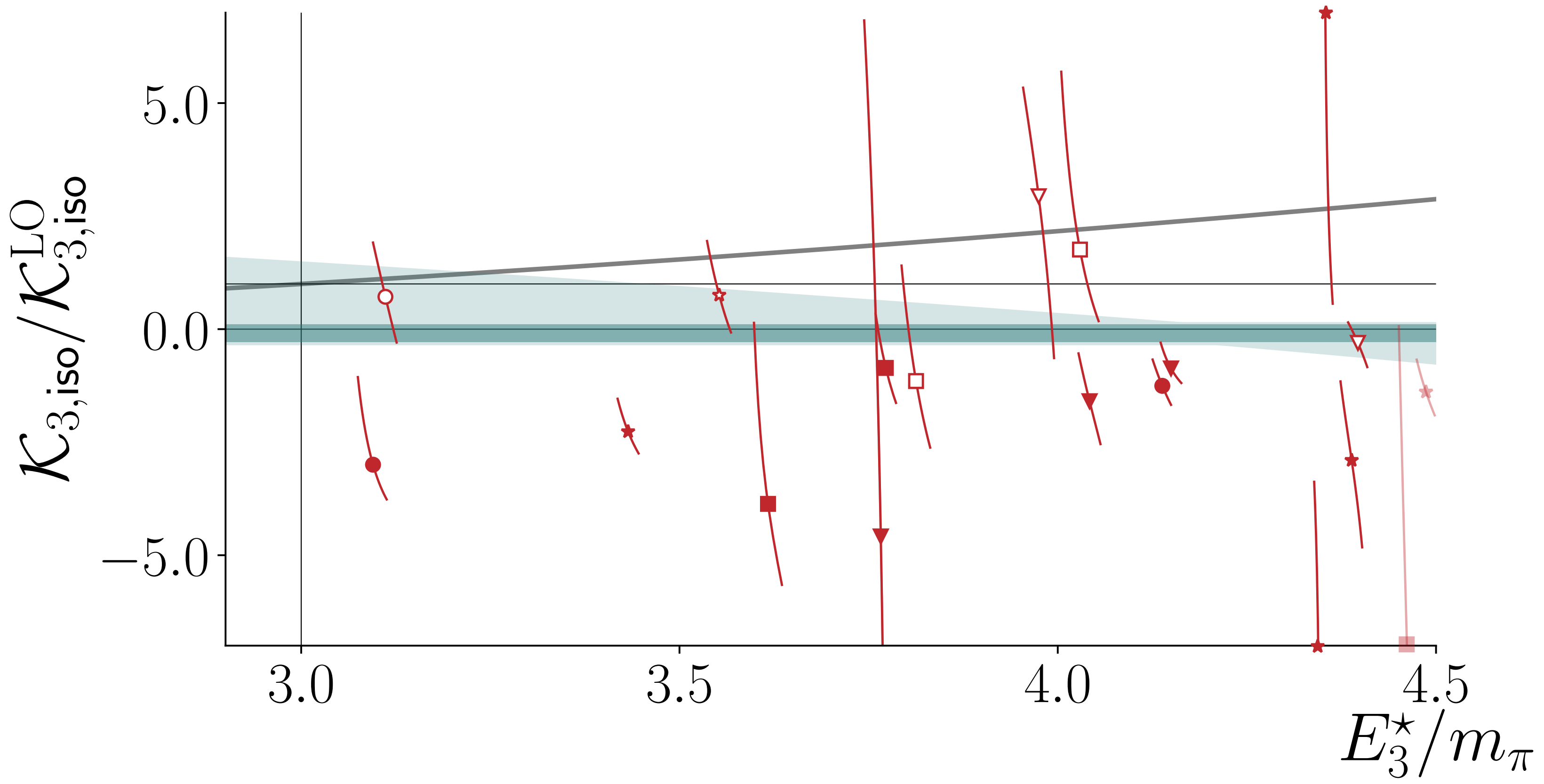
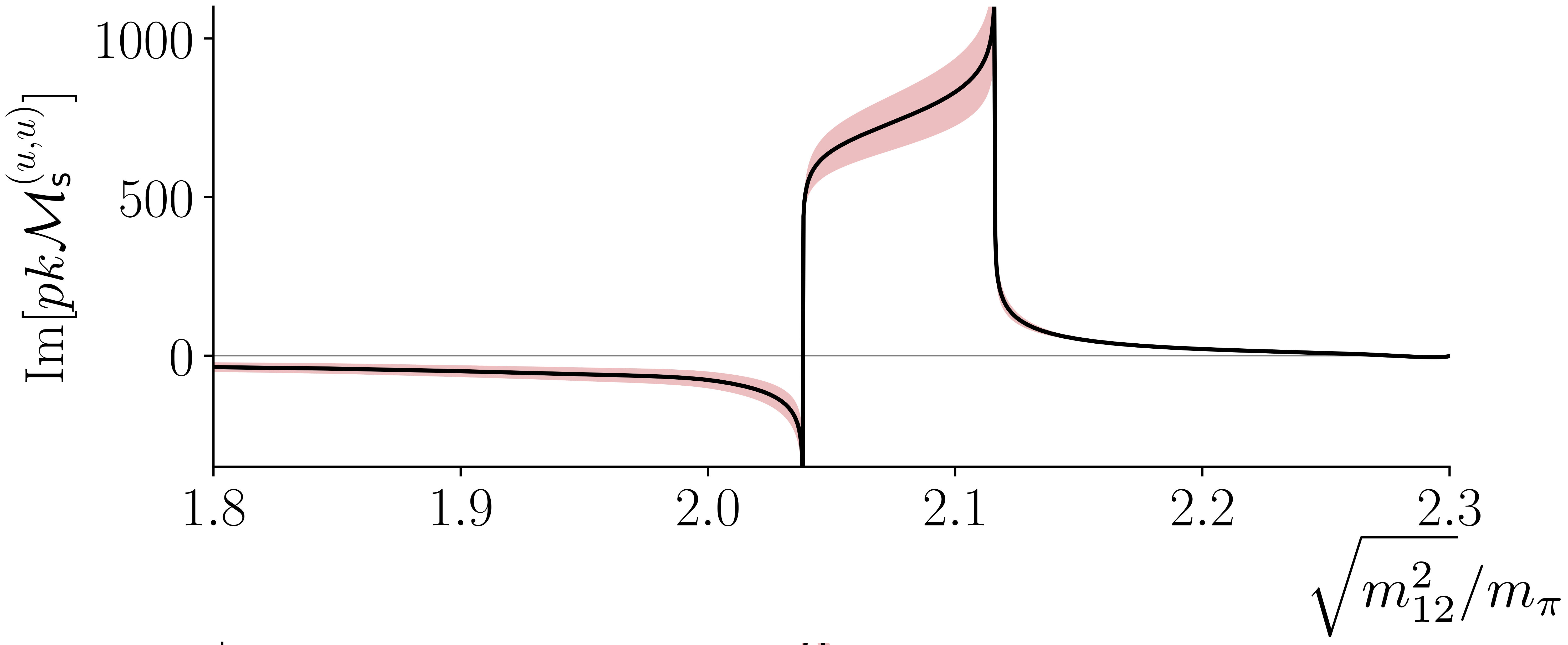
$\pi\pi\pi$ scattering

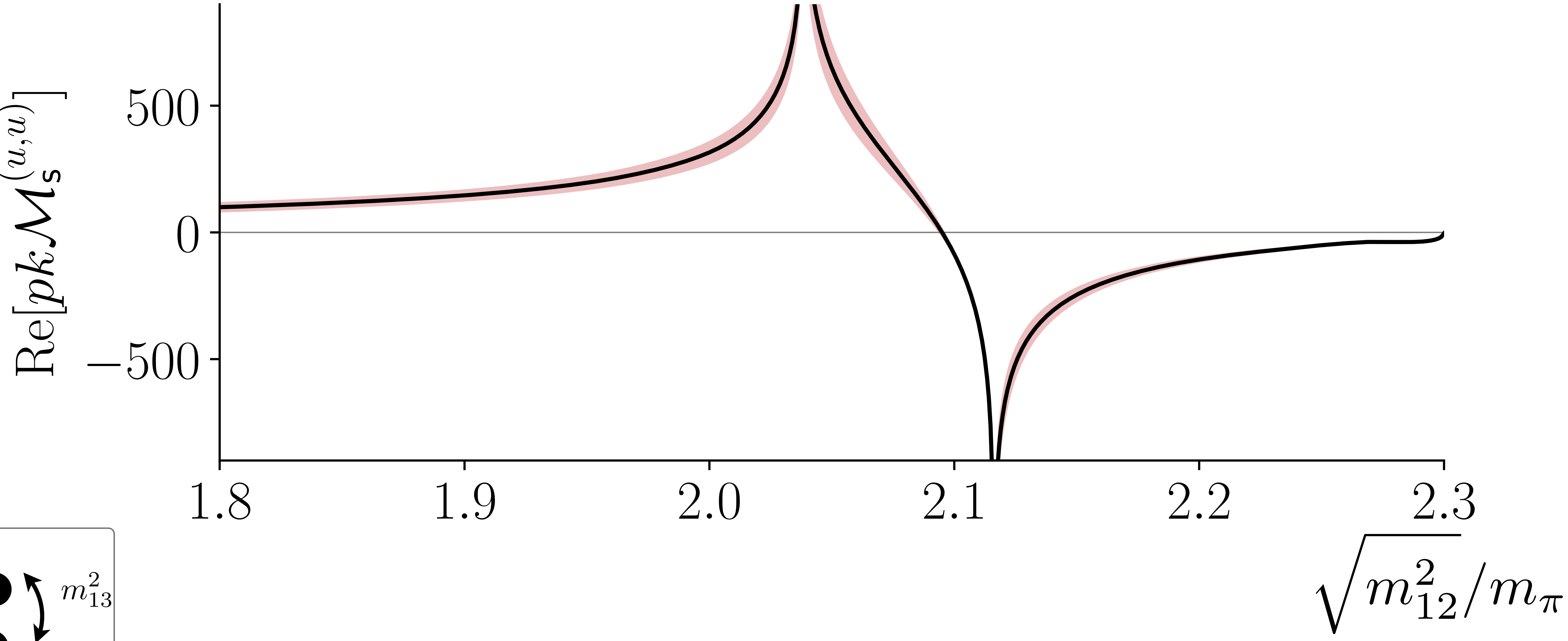
(I=3 channel, $m_\pi \sim 390\,\mathrm{MeV}$)

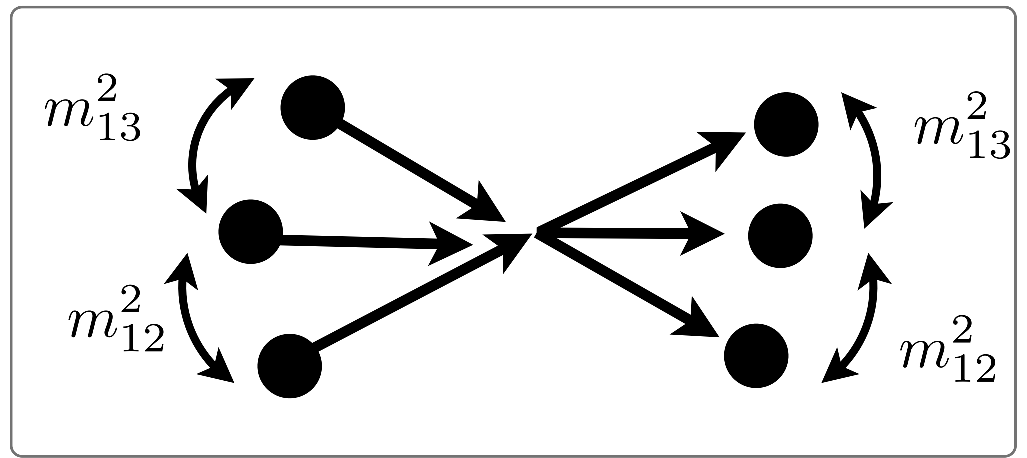
first 3body scattering amplitude from the lattice QCD!





$$i\mathcal{D} + i\mathcal{L}[\mathcal{D}] \cdot \mathcal{F}[\mathcal{D}, \mathcal{K}_3] \cdot \mathcal{R}[\mathcal{D}]$$





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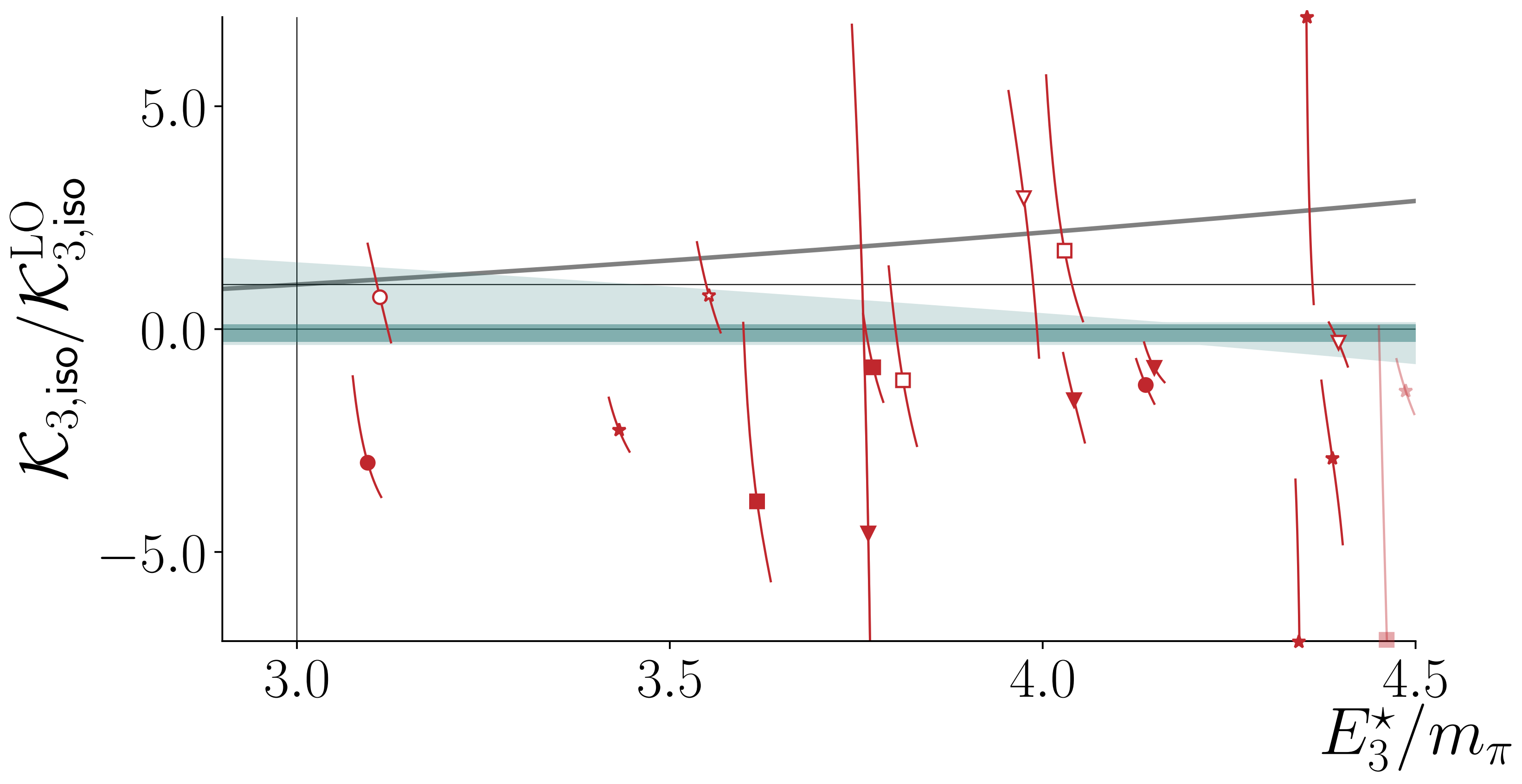
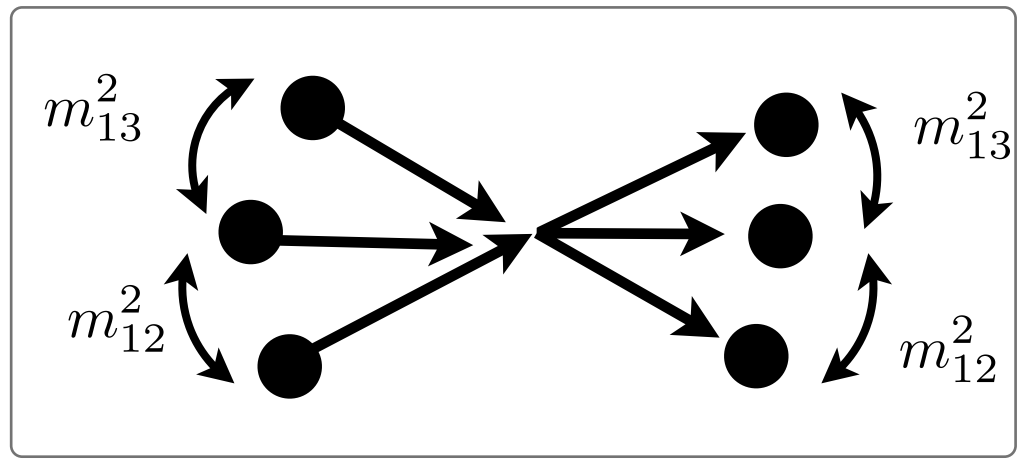
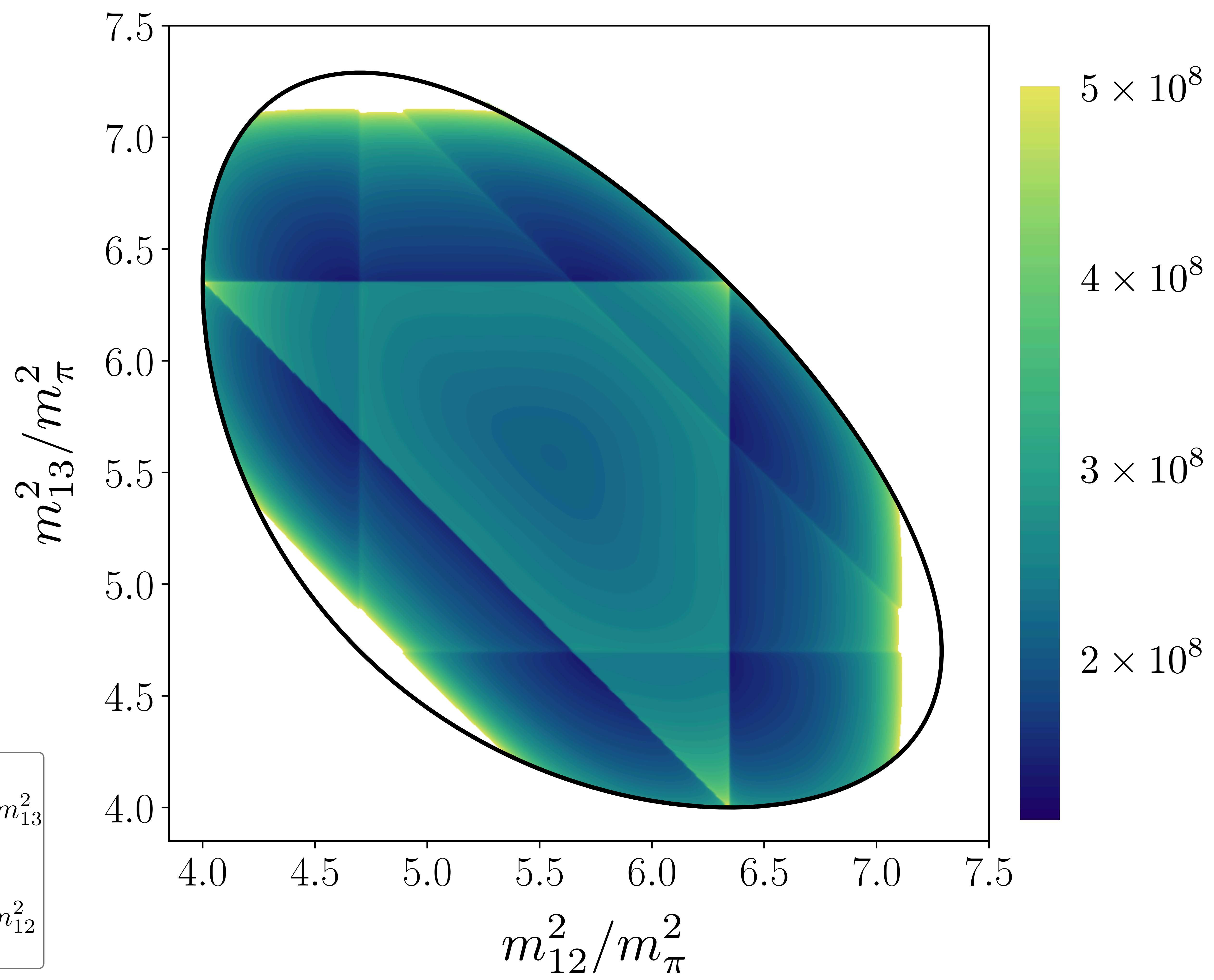


$$i\mathcal{D} + i\mathcal{L}[\mathcal{D}] \cdot \mathcal{F}[\mathcal{D}, \mathcal{K}_3] \cdot \mathcal{R}[\mathcal{D}]$$





Hansen, RB, Edwards, Thomas, & Wilson (2020)

outline

- [x] integral equations
- [x] angular momentum projection
- [x] finite-volume formalism
- [x] a lattice QCD calculation
- [ ] toy model calculations
- [ ] Efimov physics
- [ ] consistency checks and the breakdown of Lüscher

[won’t present, but happy to discuss]

Hansen, RB, Edwards, Thomas, & Wilson (2020)
Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)
Jackura, RB (2023)
RB, S. R. Costa, Jackura, (2024)
Dawid, RB, Islam, Jackura, (2023)

A very interesting example

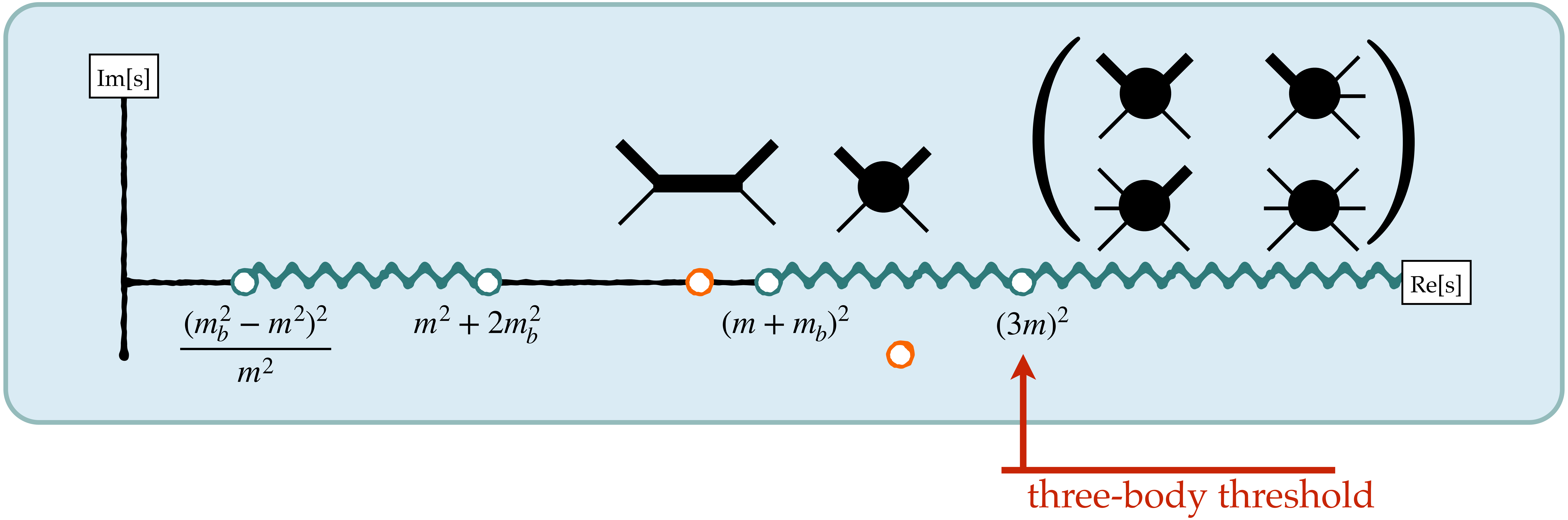
Consider a theory with a two-body bound state:

- Arguably the most singular example,
- testing formalism,
- exploring Efimov physics,
- towards nuclear physics,
- …

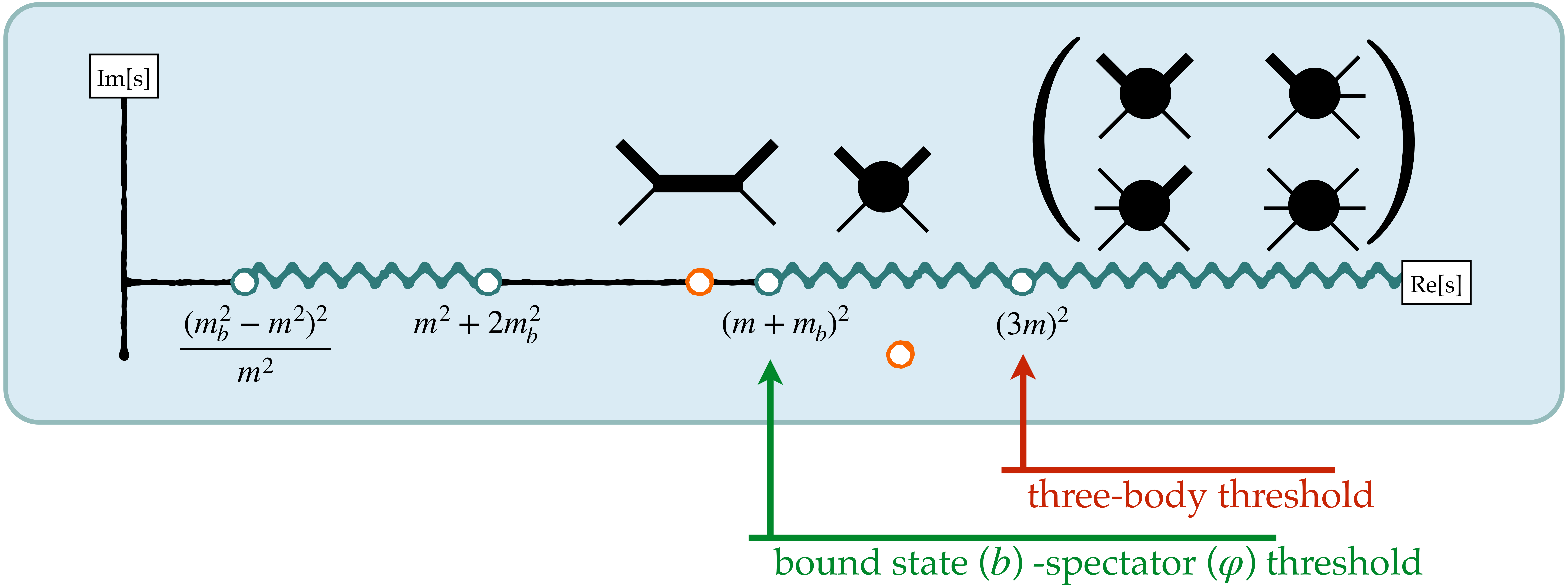
Can get a bound state using the effective range expansion at leading order:

- $\mathcal{M}_2(s) \sim \dfrac{1}{q \cot \delta(s) - iq} = \dfrac{1}{-\frac{1}{a} - iq}$
- if $a > 0$, can have a pole at $q = i\kappa = i/a$,
- bound-state mass $m_b = 2\sqrt{m^2 - 1/a^2}$,

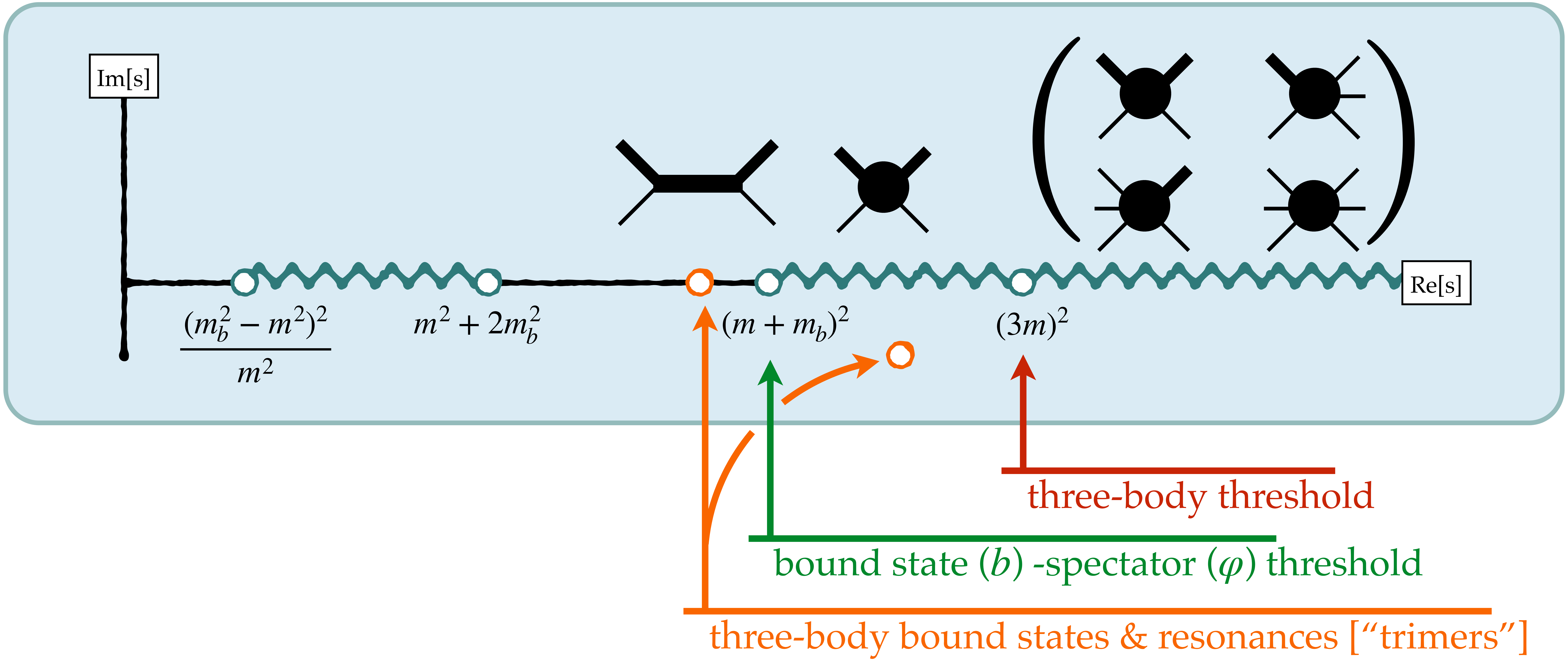
The singularity landscape



The singularity landscape



The singularity landscape



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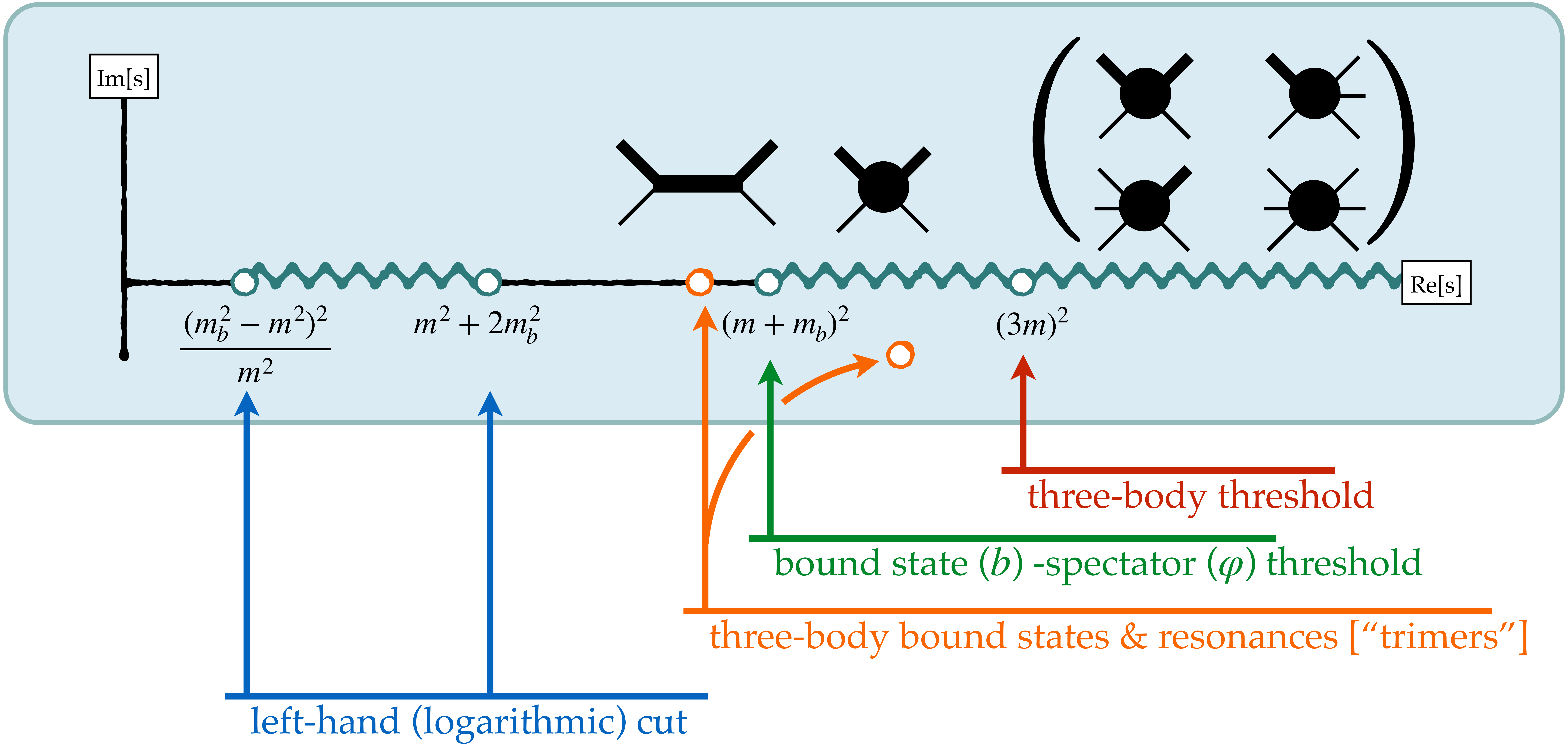
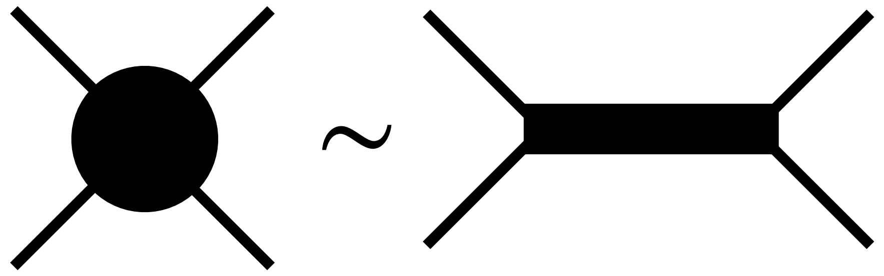


Obtaining the $b + \varphi$ amplitude

We can obtaine the $\mathcal{M}_{\varphi b}$ amplitudes using LSZ:

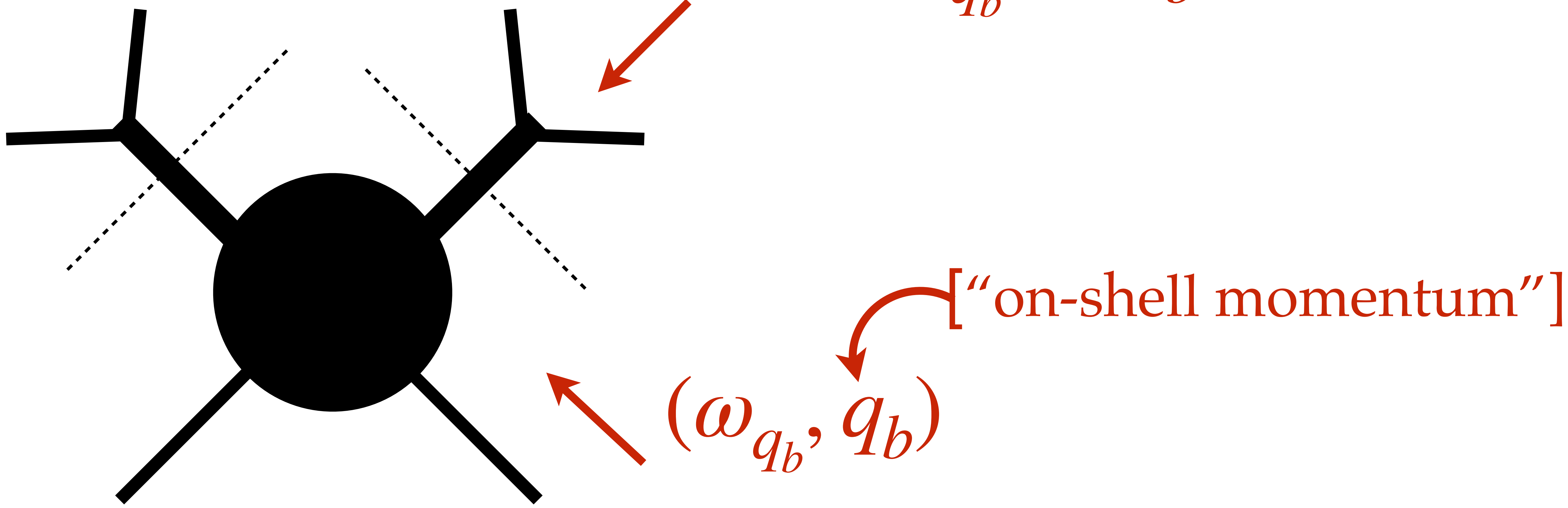
- Two-body bound state:

$\mathcal{M}_2(s) =$  $\sim \dfrac{-g^2}{s - m_b^2}$

- Bound state / spectator scattering amplitude

$$\mathcal{M}_{\varphi b}(s) = \frac{1}{\mathcal{K}^{-1}_{\varphi b}(s) - i\rho_{\varphi b}} = \lim_{\sigma_k, \sigma_p \to m_b^2} \mathcal{D}(s,k,p) \frac{(\sigma_k - m_b^2)(\sigma_p - m_b^2)}{g^2}$$

$$= \lim_{\sigma_k, \sigma_p \to m_b^2} d(s,k,p) g^2$$



Solving integral equations

- Deform contour to miss singularities and discretize momenta
  - sometimes useful / / sometimes critical

- Discretize momenta: $$d(p', s, p) = -G(p', s, p) - \int_0^{q_{\rm max}} \frac{dq\, q^2}{(2\pi)^2 \omega_q} G(p', s, q) \mathcal{M}_2(q, s)\, d(q, s, p)$$
$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\rm max}} K(p', s, q)\, d(q, s, p)$$

[contains pole, logarithmic and **square root cuts]**

Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)

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- Use linear algebra:

$$[1 + \boldsymbol{K}] \cdot \vec{d}_{\mathbf{sol}}(s, p) = -\vec{G}(s, p)$$

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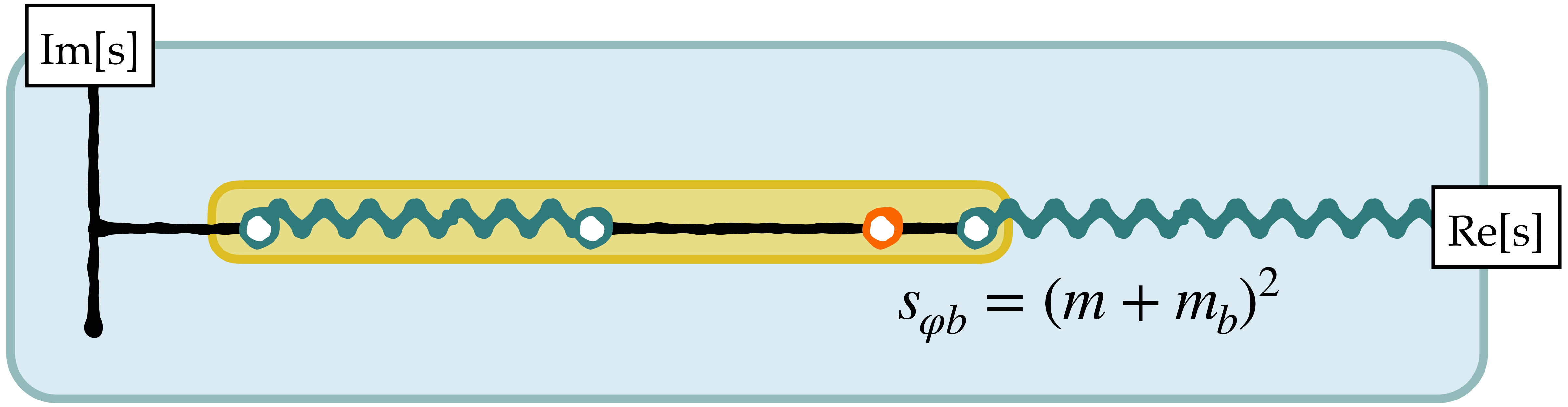
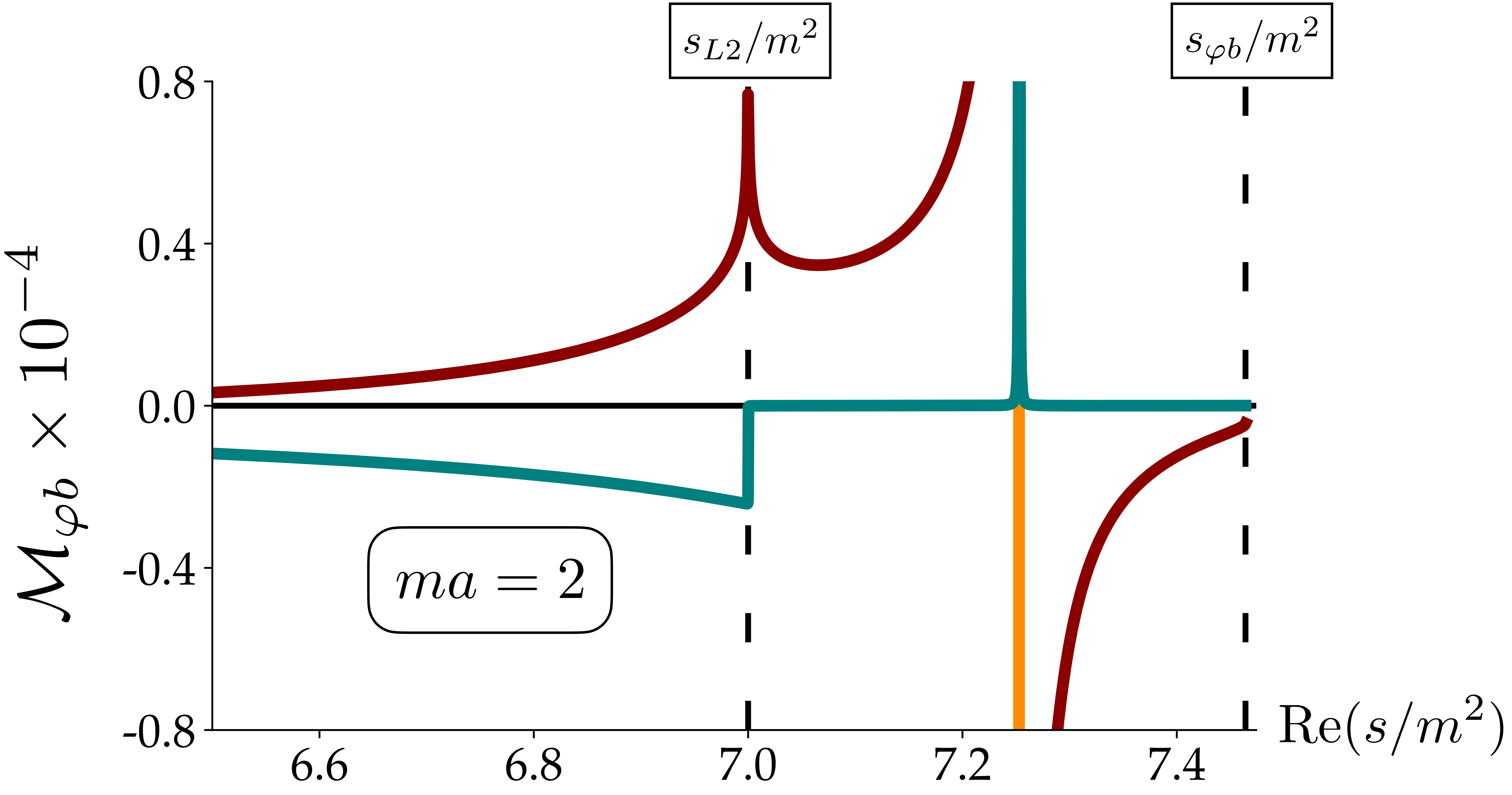
$$\approx -G(p', s, p) - \sum_{q=0}^{q_{\max}} K(p', s, q)\, d(q, s, p)$$
- Use linear algebra:
$$[1 + \boldsymbol{K}] \cdot \vec{d}_{\mathbf{sol}}(s, p) = -\vec{G}(s, p)$$
- Use integral equation to interpolate or extrapolate:
$$d(p', s, p) \approx -G(p', s, p) - \vec{K}(p', s) \cdot \vec{d}_{\mathrm{sol}}(s, p)$$

Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)

Some amplitude results





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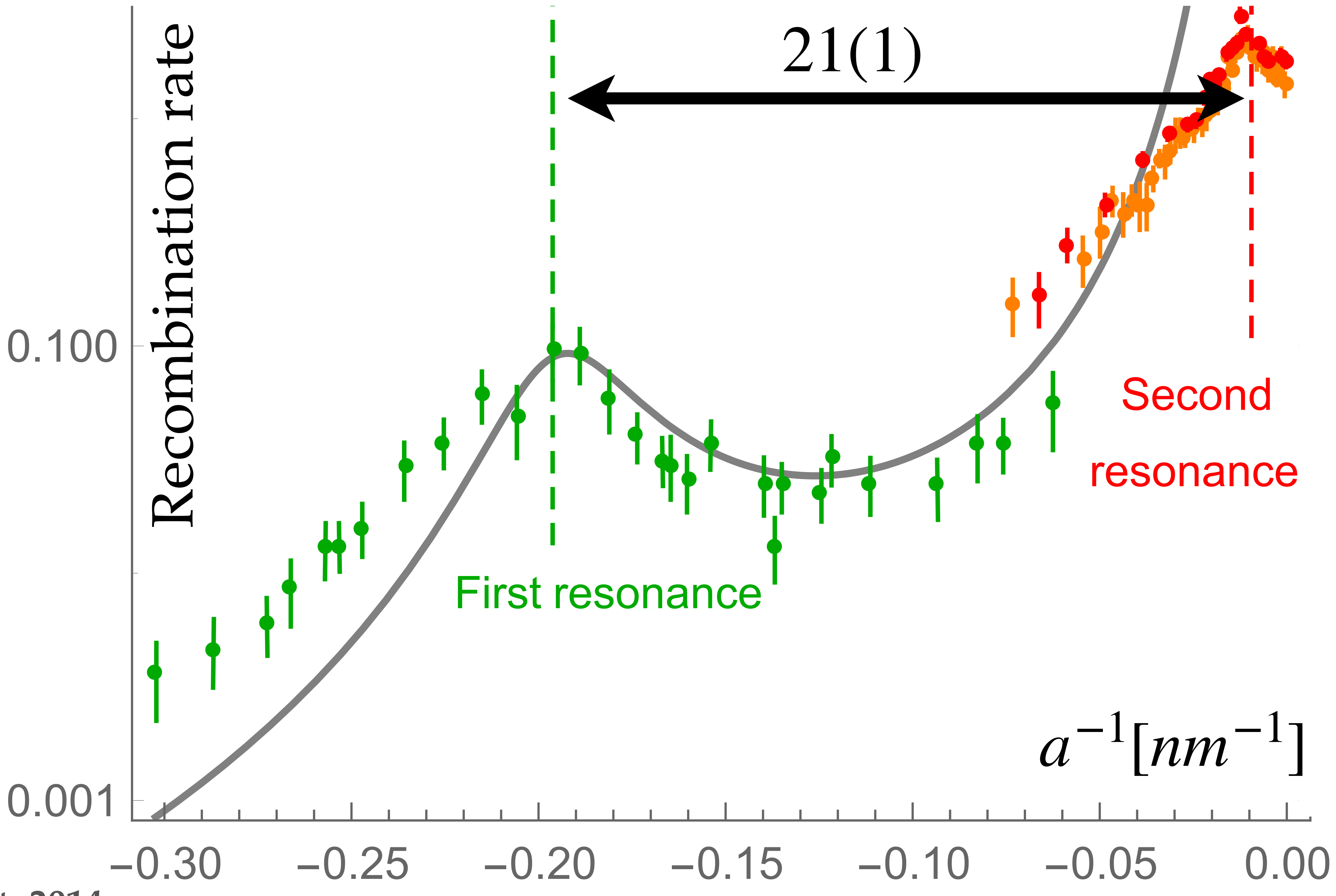
Dawid, RB, Islam, Jackura, (2023)

Efimov physics

Unitary limit: $p\cot\delta = -\frac{1}{a} + \frac{rp^2}{2} + \cdots = 0$

Pole in the two-body scattering amplitude at threshold: $\mathcal{M} \sim \frac{1}{p\cot\delta - ip} = \frac{1}{ip}$

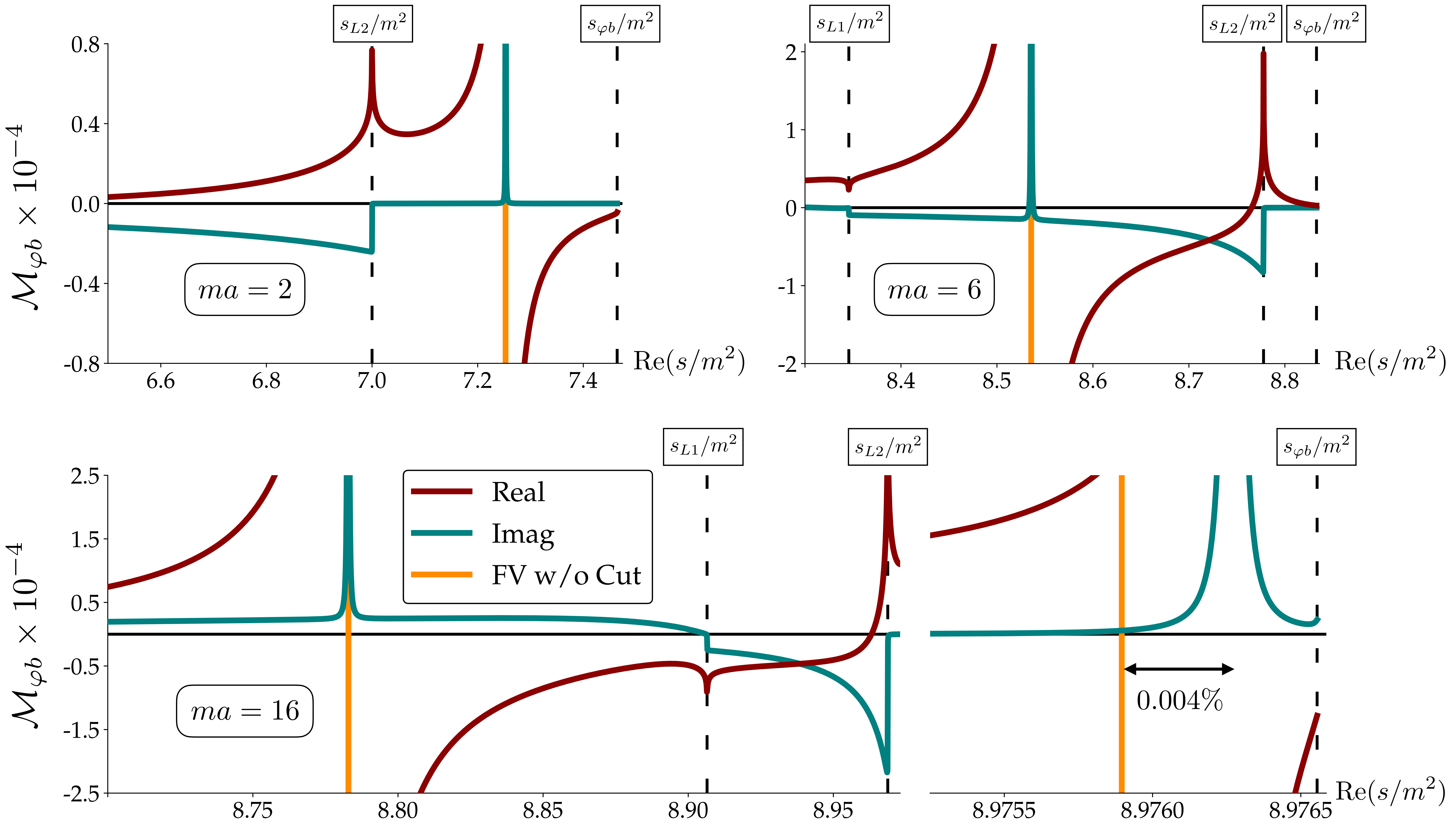
Infinite tower of geometrically-separated three-body bound states: $E_{N+1} = E_N/\lambda^2$ where $\lambda = 22.69438$



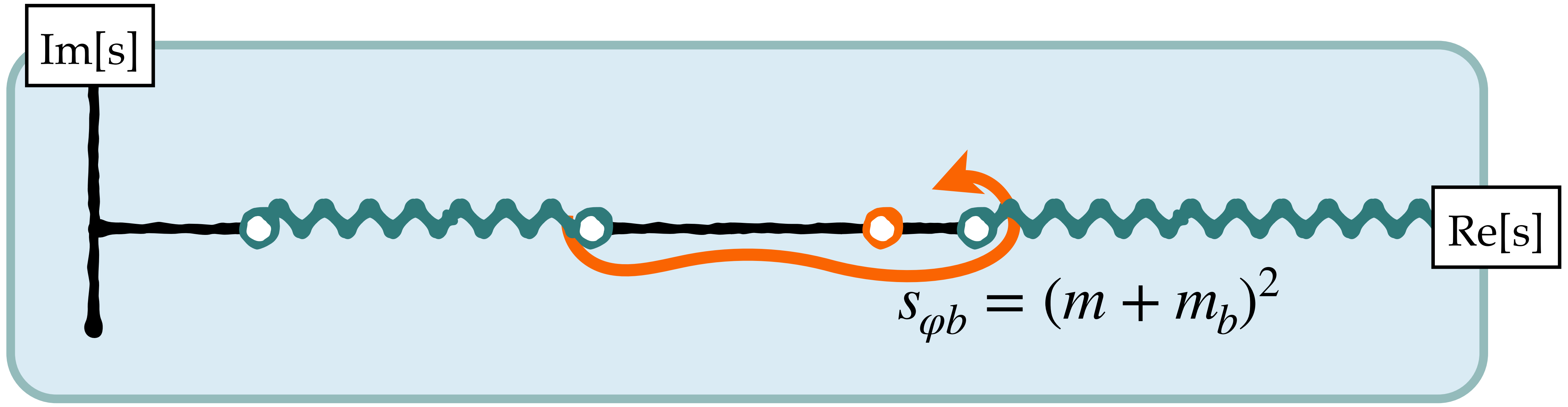
Phys. Rev. Lett. **2014**

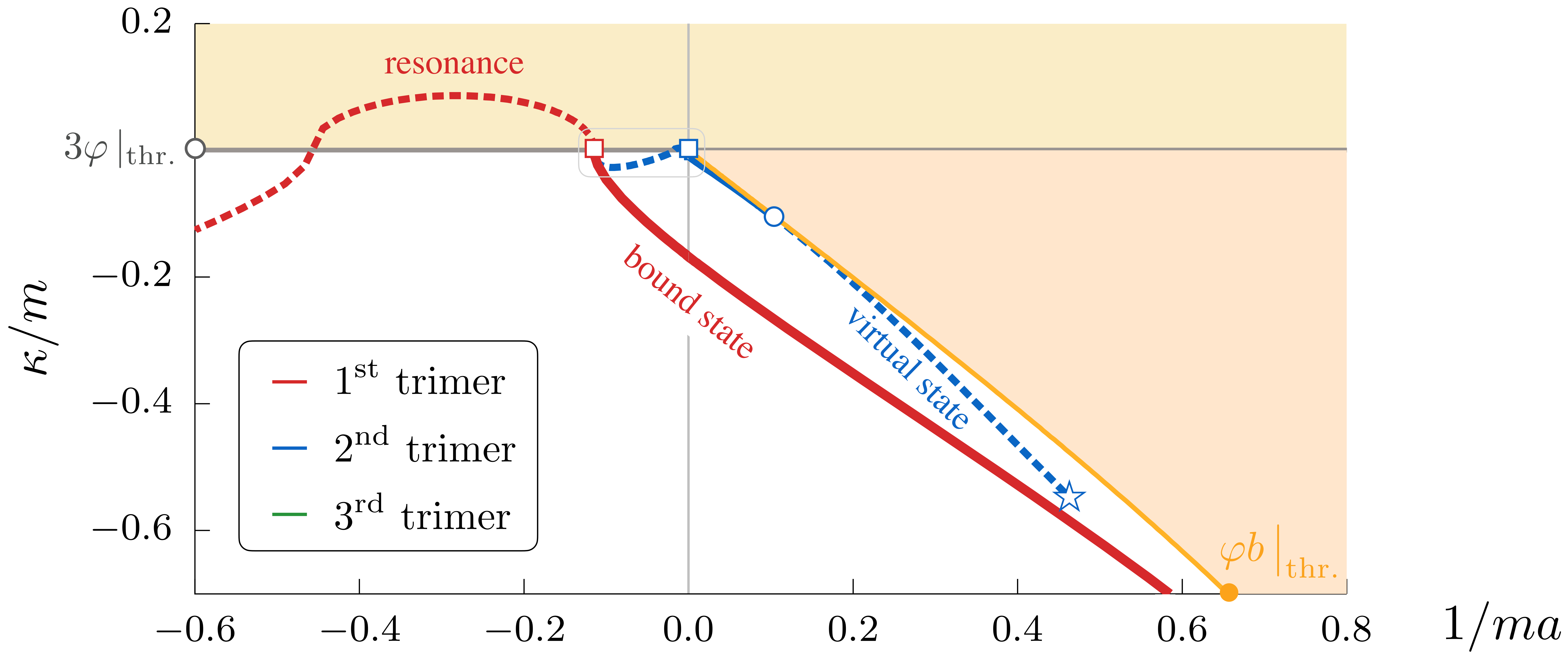

Vitaly Efimov

increasing a



the *Efimov evolution*





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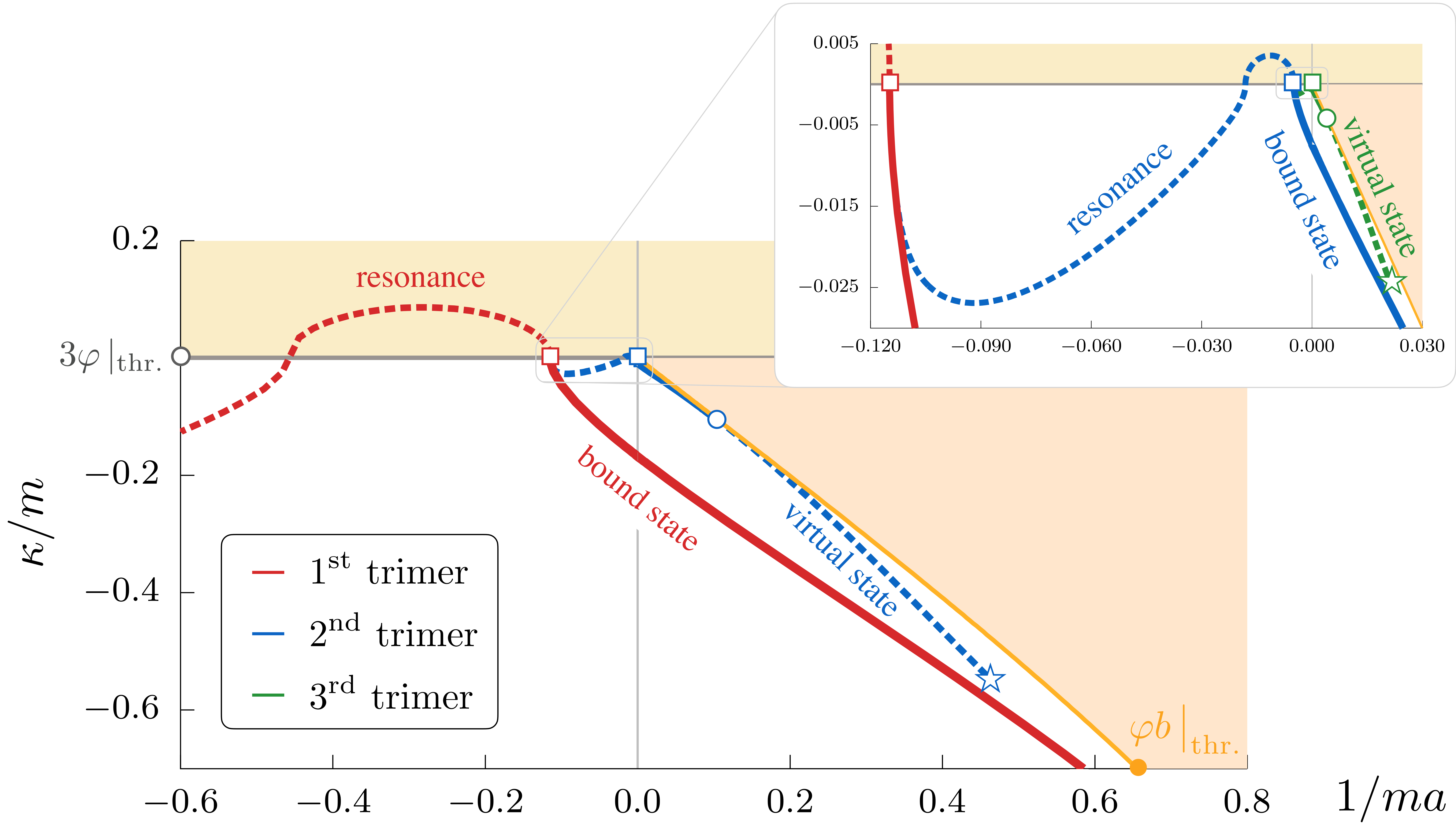

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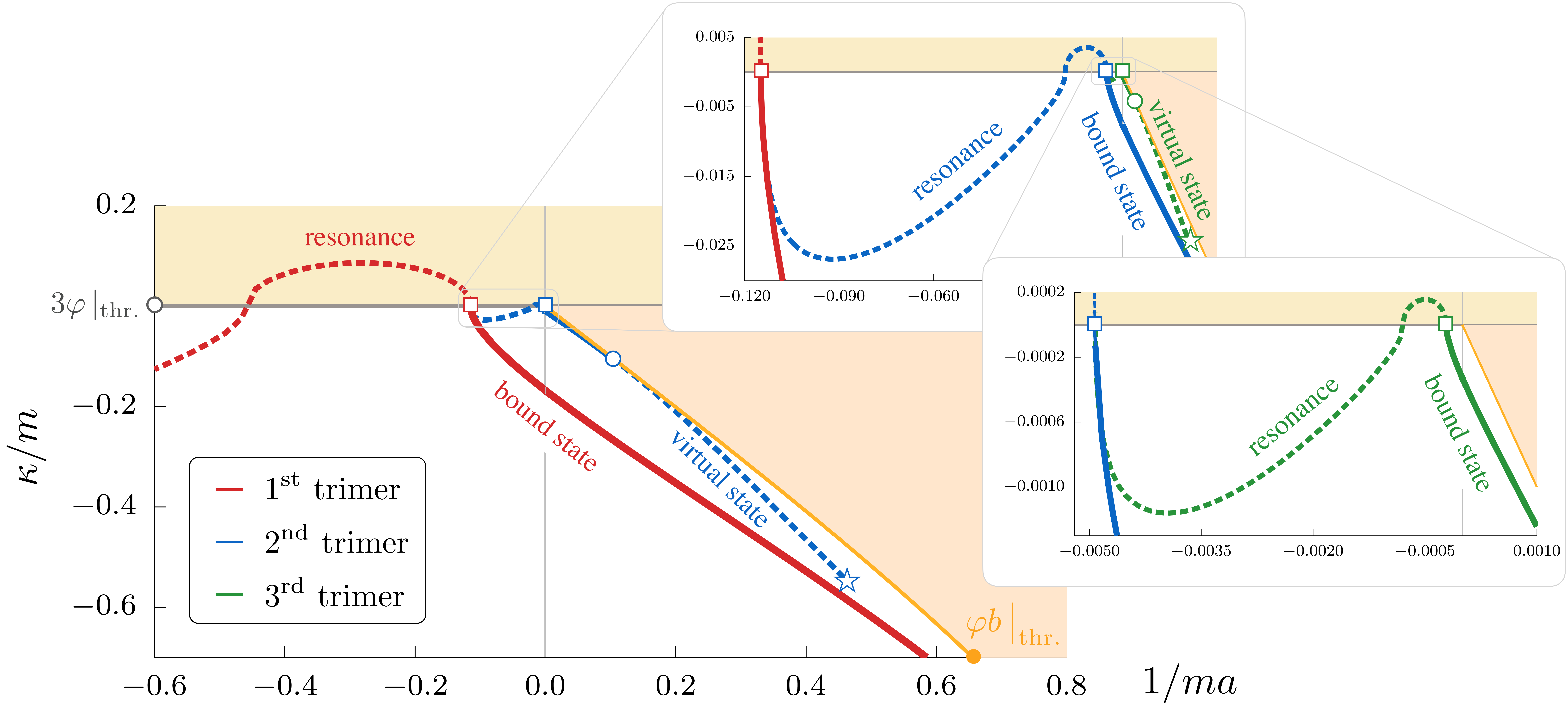
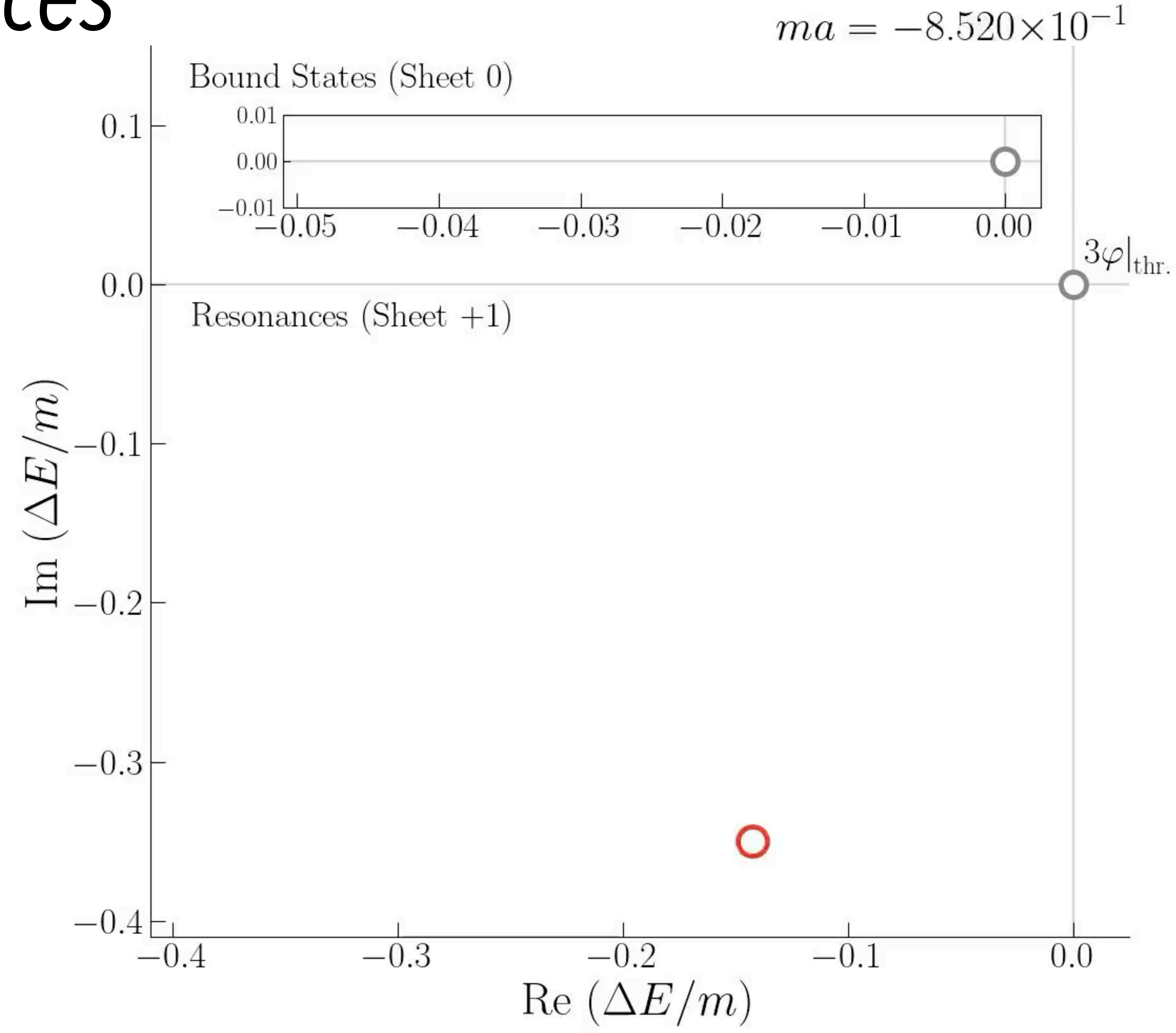


the *Efimov resonances*



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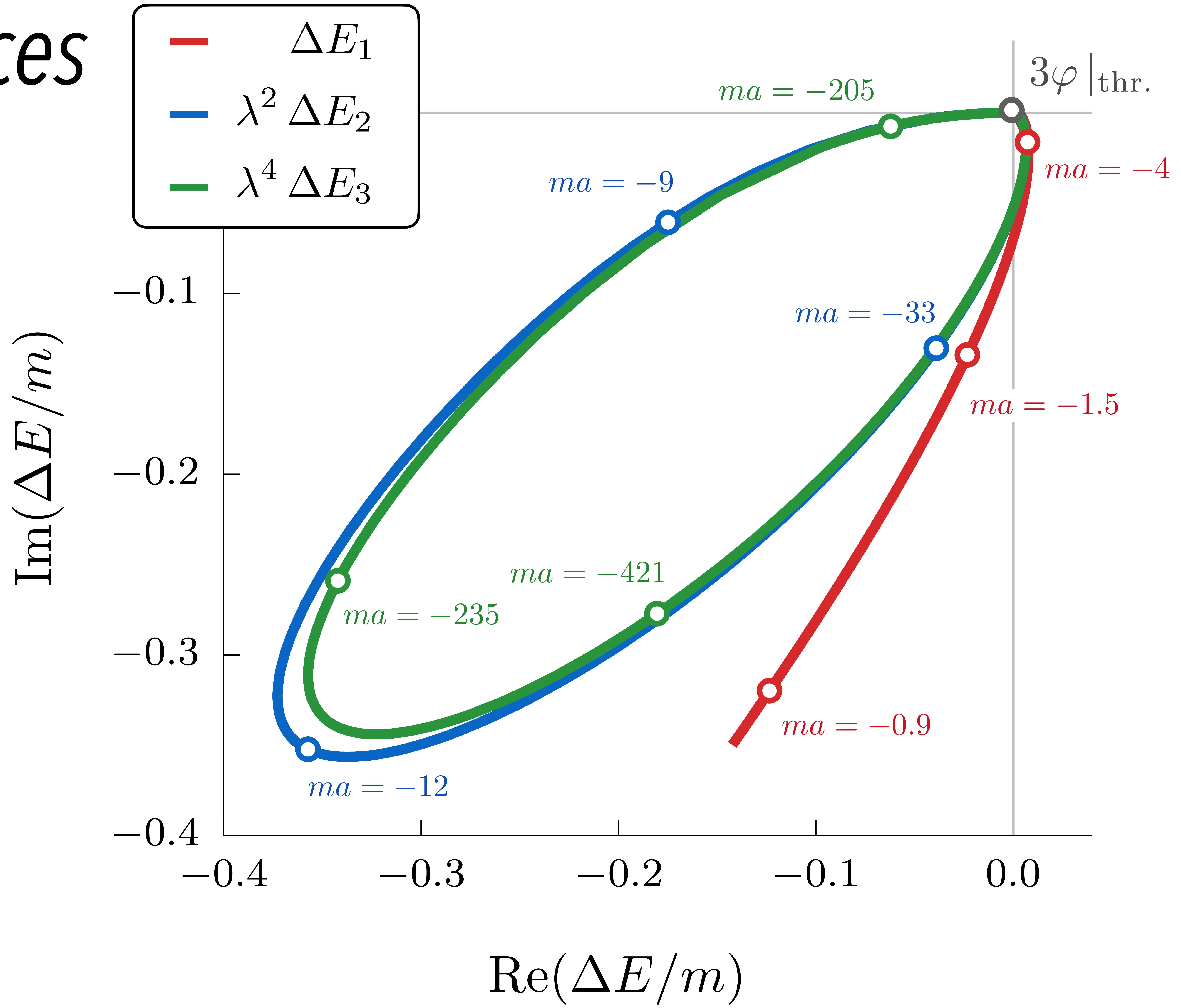


the conjecture, the puzzle, & the quirky


the conjecture

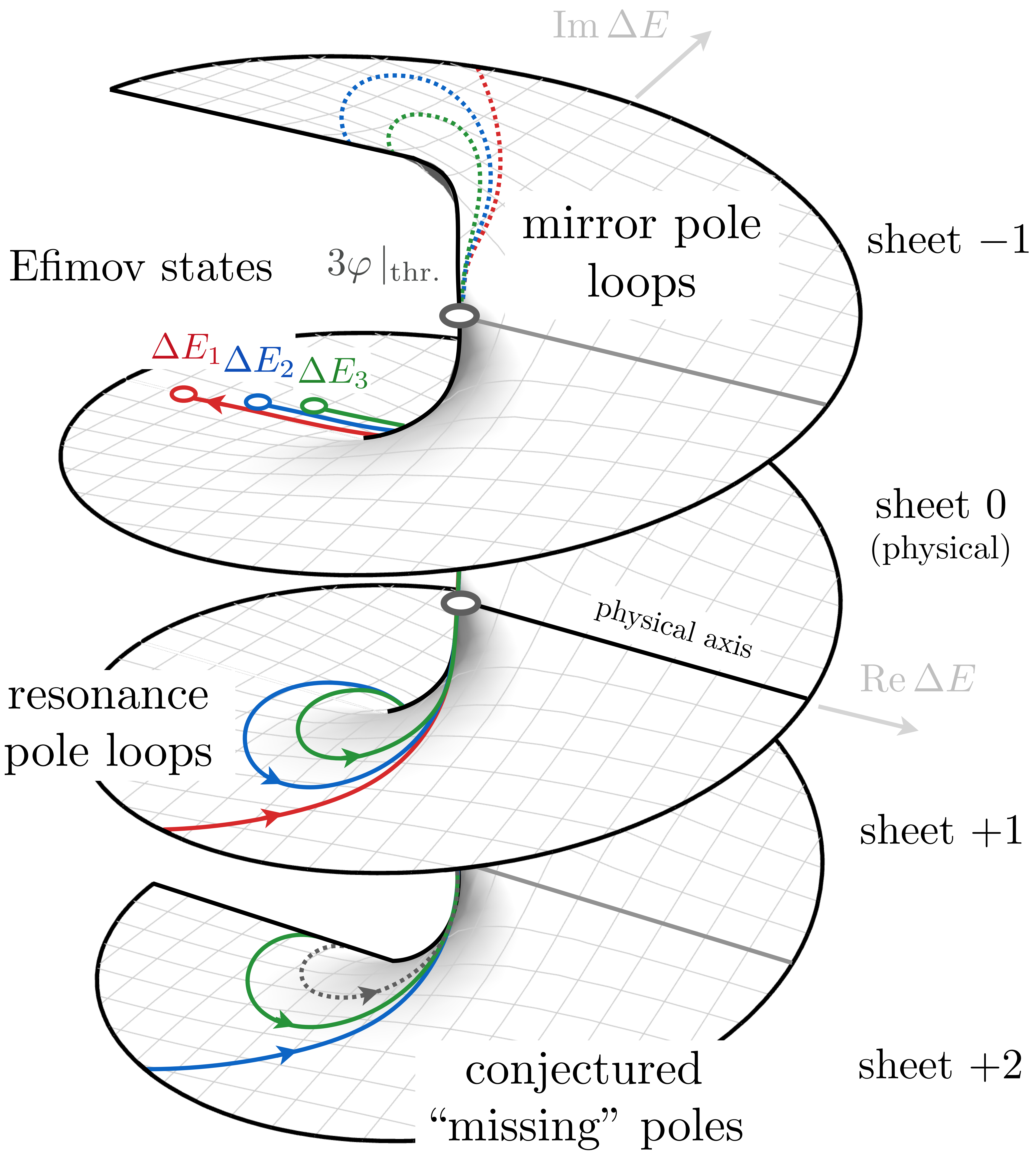

the puzzle


the quirky

the conjecture *the Efimov elevator*

Although we do not see beyond the nearest sheets…we conjecture that Efimov resonances are marching in synchronous from the infinite number of sheets.

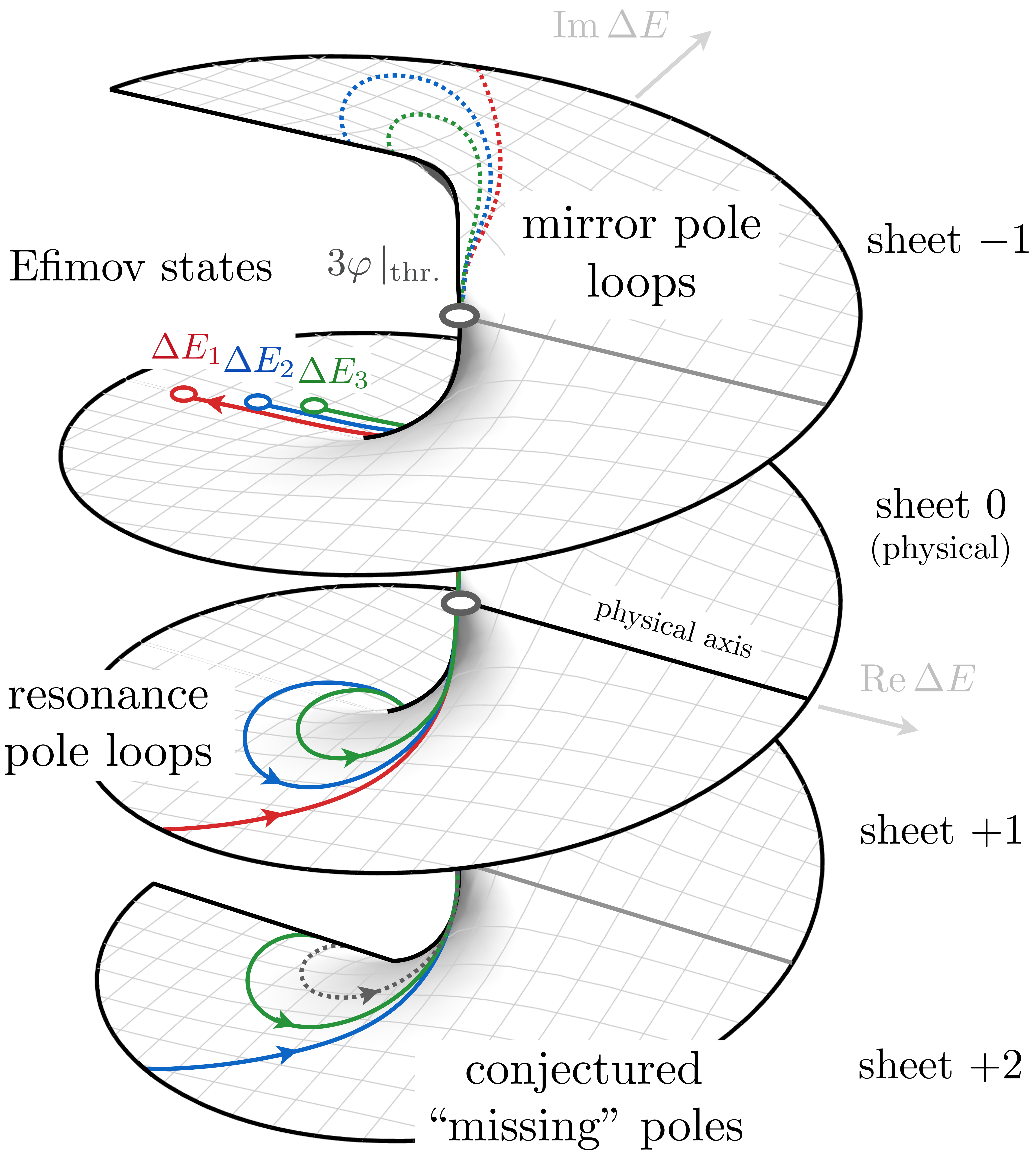
When the n^{th} state reaches the j^{th} sheet, the $n{+}1^{th}$ one reaches the $j{+}1^{th}$ sheet.



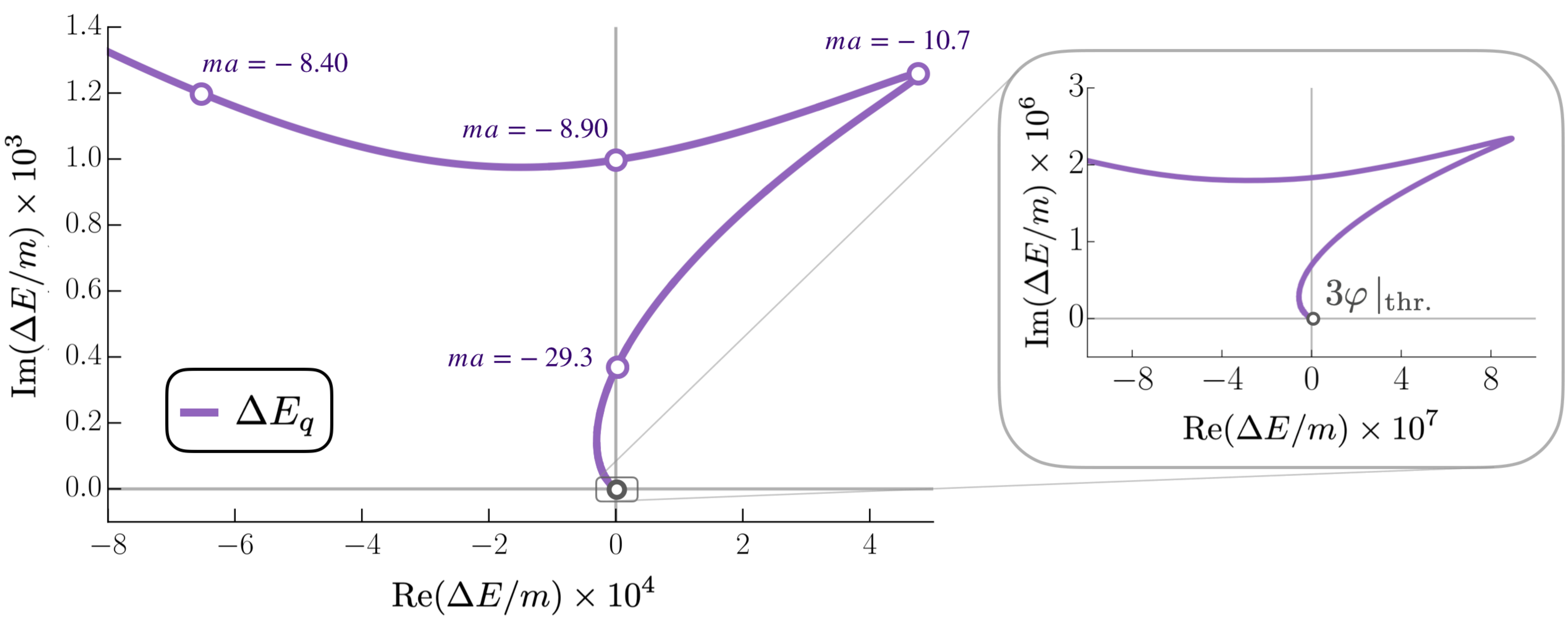
the puzzle

What happens are the "ground floor" of the Efimov elevator?

Two poles come in: one from the +1 sheet and another from the -1 sheet…but only one pole appears in sheet 0?



the quirky



ma = − 8.40
ma = − 8.90
ma = − 10.7
ma = − 29.3
ΔE_q
Im(ΔE/m) × 10^3
Re(ΔE/m) × 10^4
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−8
−6
−4
−2
0
2
4
3φ |thr.
Im(ΔE/m) × 10^6
Re(ΔE/m) × 10^7
3
2
1
0
−8
−4
0
4
8

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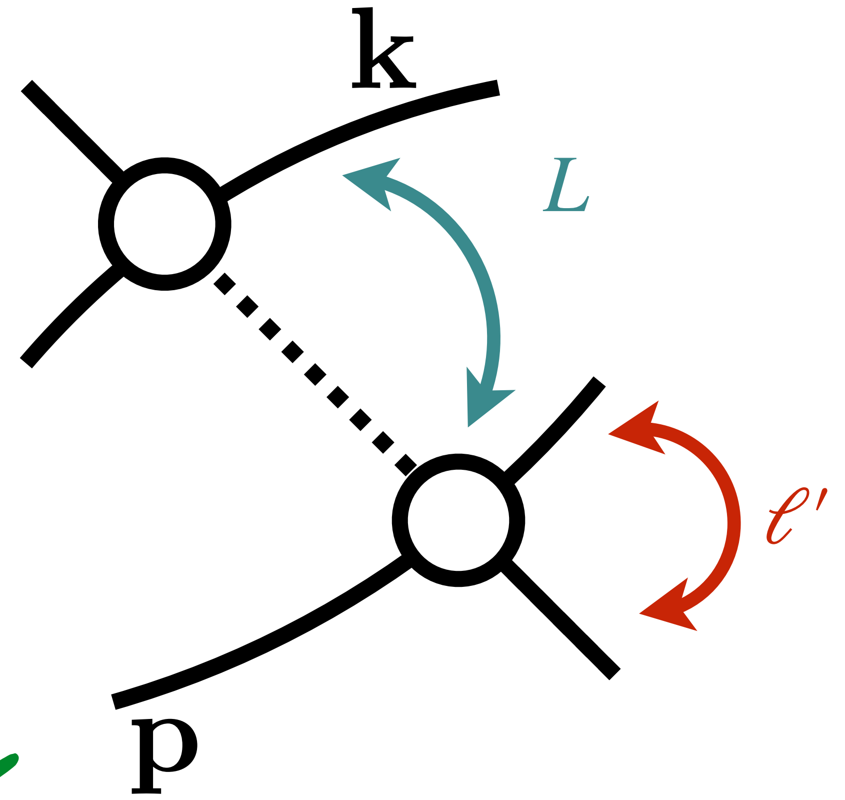
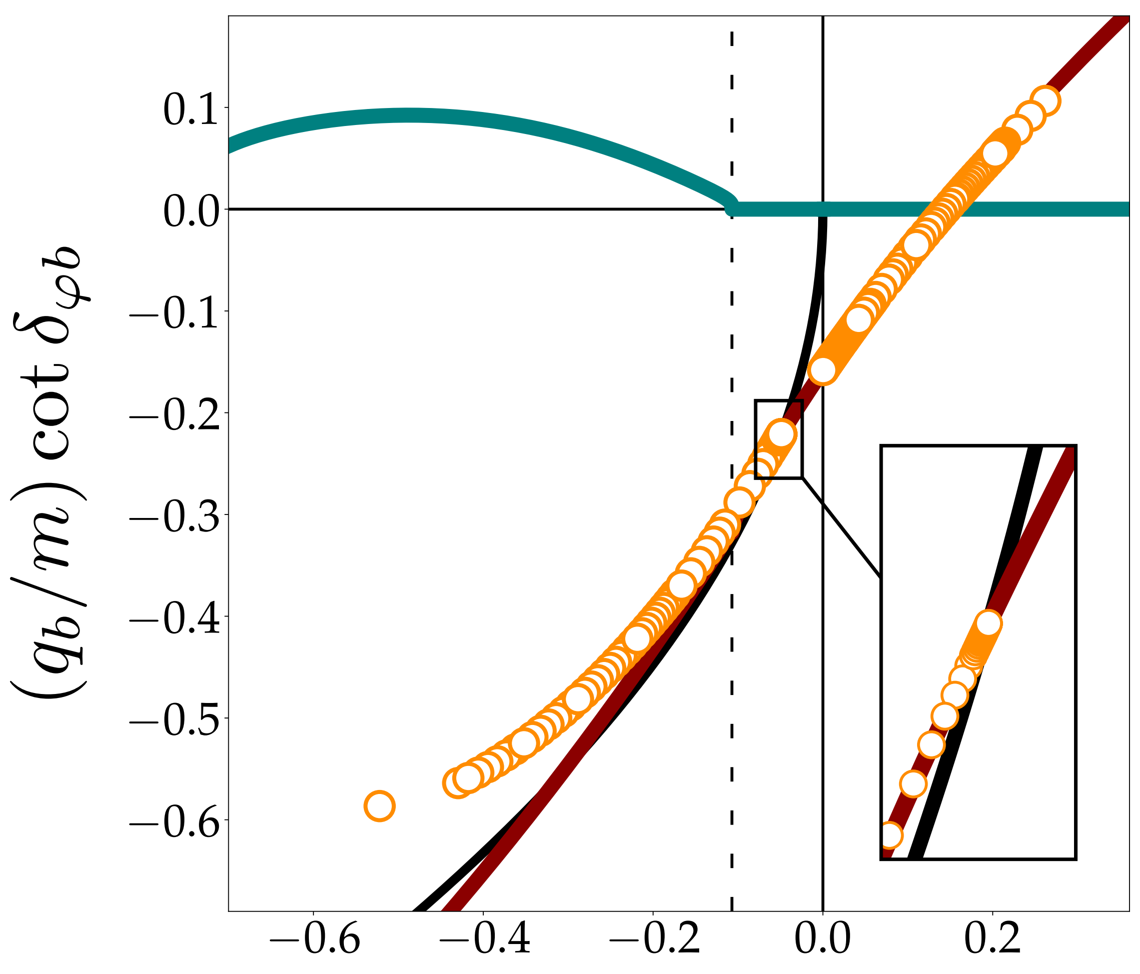
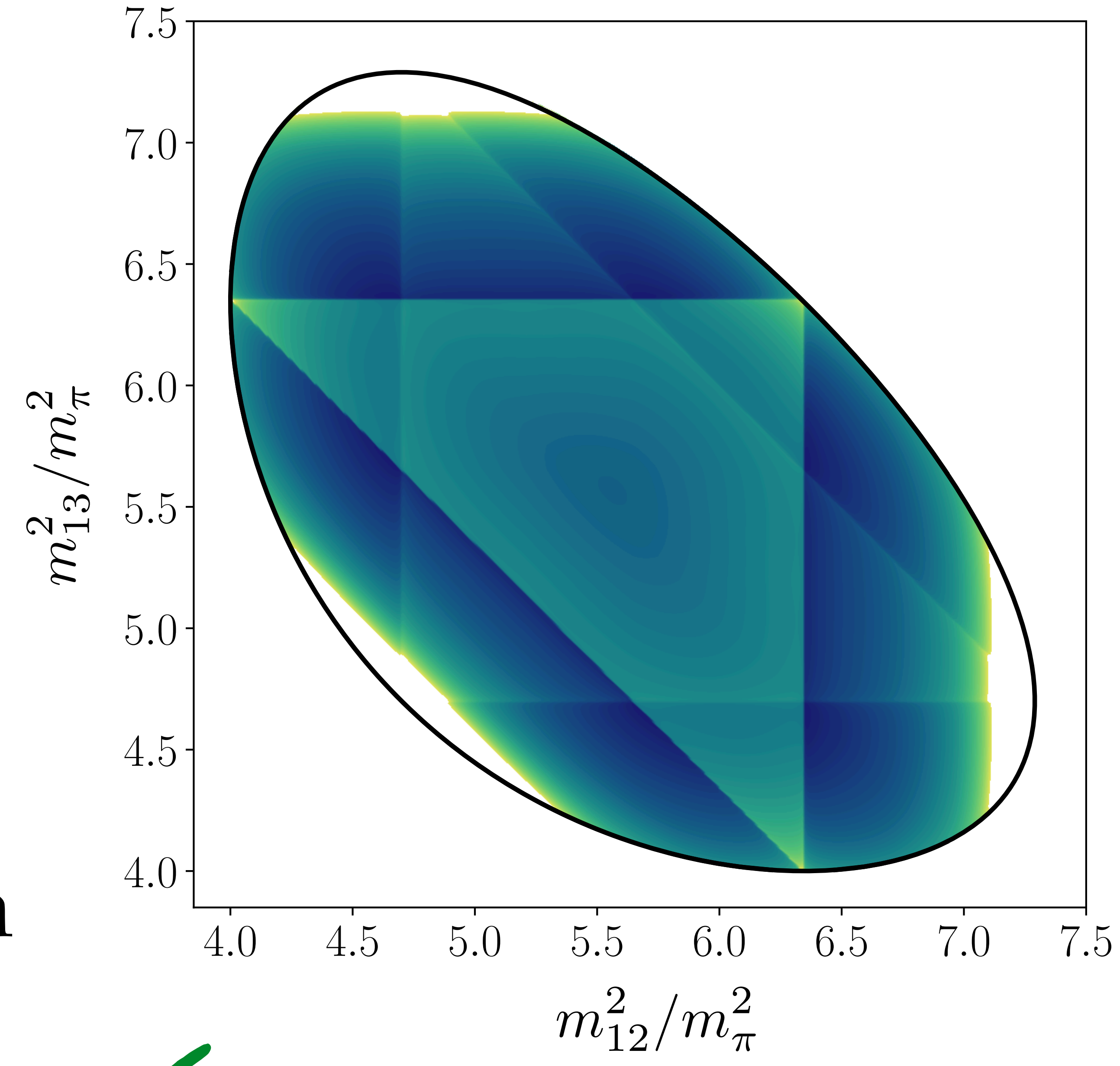
rapidly developing field!



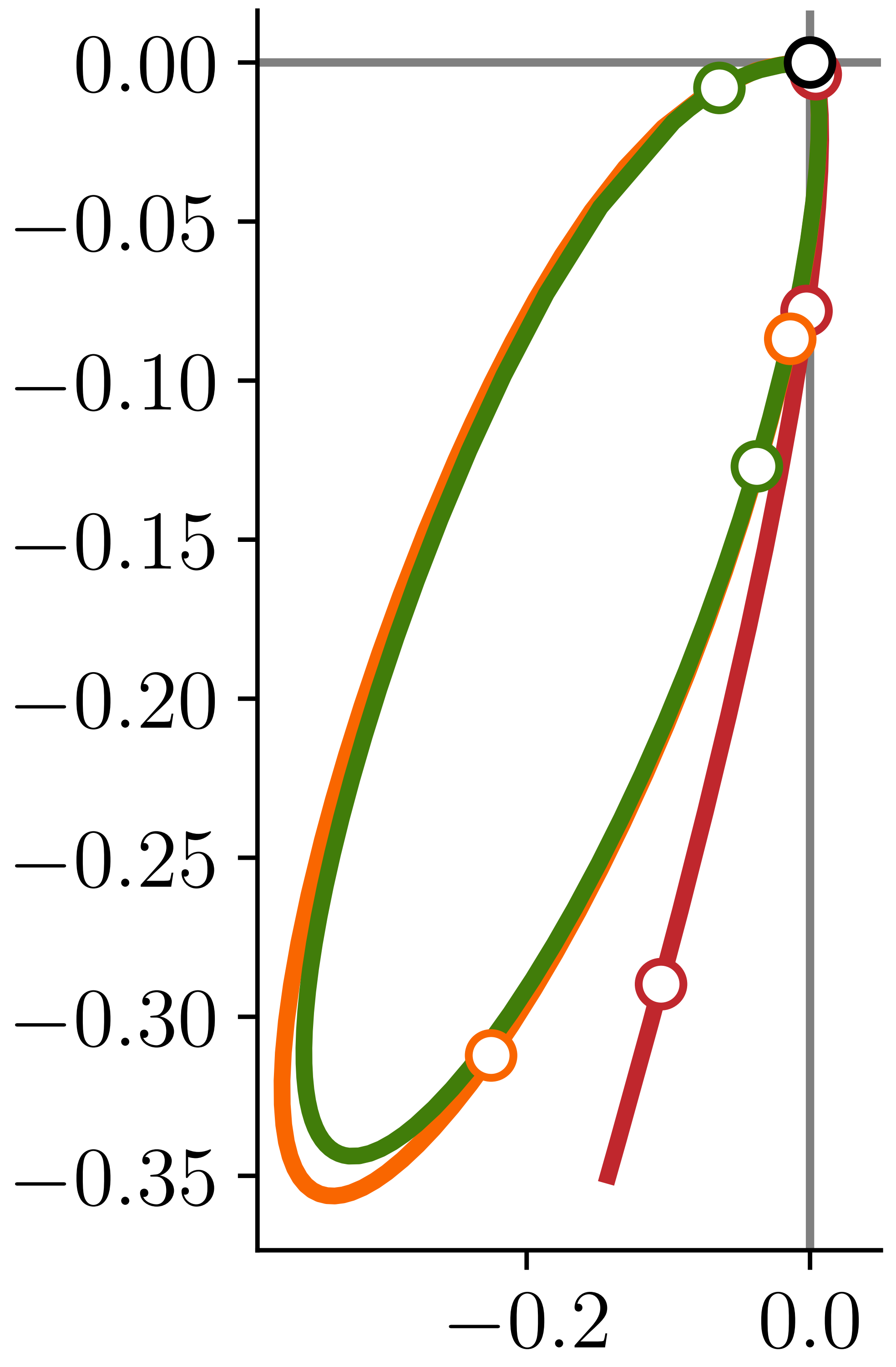
formal developments



checks of the formalism



actual lattice calculations



further explorations

Romero-López, Sharpe, Blanton, RB, & Hansen (2019)
Hansen, RB, Edwards, Thomas, & Wilson (2020)
Jackura, RB, Dawid, Islam, & McCarty (2020)
Dawid, Islam, & RB (2023)
Jackura, RB (to appear)
Dawid, RB, Islam, Jackura, (2023)