Some Non-Perturbative Aspects of 2+1 D and 3+1 D Gauge Theories

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We'll explain some recent non-perturbative results about gauge theories in 3 dimensions and emphasize the connection with gauge theories in 4 dimensions.

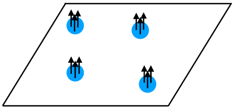
The presentation is mostly based on collaborations (2017-2018) with Davide Gaiotto, Jaume Gomis, Nathan Seiberg, as well as work in progress.

Terminology

- **Trivial Gapped**: No massless excitations, no topological theory, trivial (product) wave function
- **Topological Field Theory (TFT)**: no massless excitations, but some long range entanglement and topological order (such as anyons). Nontrivial ground state wave function.
- **Massless phases**: This could be due to a Conformal Field Theory, or due to Nambu-Goldstone particles.

There could be several vacua (superselection sectors) furnishing the various types above.

QCD in 3 dimensions is conceptually different from QCD in 4 dimensions because the quarks may carry magnetic flux attachment. This is the due to the Chern-Simons term that we may add to the Lagrangian. Many dynamical effects are driven by this particular feature. The Chern-Simons term is unavoidable, since intergrating out fermions generates it.



We will concentrate our on the following model :

- N_f fundamental *complex* fermions λ_α coupled to $SU(N)$ gauge fields with a Chern-Simons term at level k. We refer to this theory simply as "QCD_3."

This model has various weakly coupled limits where we can understand the dynamics in detail. There are also strongly coupled regimes, where we have made some conjectures.

Typically, 't Hooft anomaly matching conditions (along with many other constraints) are an important ingredient in our study of strongly coupled theories. One may be worried that since we are studying 3 dimensional theories, this important tool will not be available to us.

However, recent developments uncovered a swath of new discrete anomalies.

- If a [fermionic] QFT in 2+1 dimensions has a time reversal symmetry with $T^2 = (-1)^F$, there is a possible obstruction to studying this theory on a non-orientable space. The obstruction is valued in $\mathbb{Z}_{16}$.
- Mixed 't Hooft anomalies between time reversal symmetry and global symmetries. 't Hooft anomalies for global symmetries.

Matching these discrete anomalies is, for practical purposes, as constraining and non-trivial as matching continuous 't Hooft anomalies.

The Lagrangian is

$$\mathcal{L} = \frac{-1}{4g^2}\mathrm{Tr}F^2 + \frac{k}{4\pi}\mathrm{Tr}\left(AdA + \frac{2}{3}A^3\right) + \mathcal{L}_{matter} \ .$$

$$\mathcal{L}_{matter} = \sum_{k=1}^{N_f}\left(i\bar{\psi}^k \not{D}\psi_k + \frac{m}{4\pi}\bar{\psi}^k\psi_k\right)$$

Note that m is a real parameter. Global Symmetry of the Model: $U(N_f)$. If $m = k = 0$ then we further have time reversal symmetry, with $T^2 = (-1)^F$.

Consistency requires

$$\frac{N_f}{2} + k \in \mathbb{Z} \ .$$

Decoupling Limits: $|m| \gg g^2$ **or** $k \gg N$**.**

In some corners of the parameter space the theory becomes weakly coupled.

If $|m| \gg g^2$ the quarks decouple even before the interactions set in. One has to be careful integrating them out as there is a non-decoupling effect [Redlich, Niemi-Semenoff] proportional to $m/|m| = sgn(m)$.
This shifts k according to

$$k \to k + sgn(m)\frac{N_f}{2} \ .$$

Another weak coupling limit is

$$k \gg N$$

The gauge field A now has a mass kg^2 and therefore it decouples before the interactions set in:

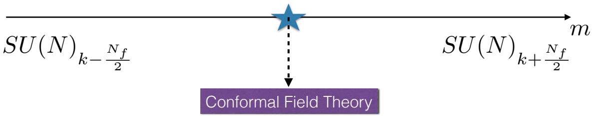
$$kg^2 \gg g^2 N \ .$$

The remaining light fields are weakly interacting and there is a weakly coupled *Conformal Field Theory* if we tune m.

Therefore the physics of large m for any k is a (possibly trivial) TFT. For large k there is a second order phase transition as we vary m, described by some (weakly coupled) Chern-Simons matter theory. On the two sides of the transition we encounter different TFTs so these transitions are typically non-Landau-Ginzburg.

We are therefore led to the following phase diagram for $k \gg 1$:

$$SU(N) + N_f\psi, \quad \text{level } k \qquad\qquad k \gg 1$$



The interesting question is where does this picture break down. This is analogous to asking where is the end of the conformal window. Surprisingly, in 3d there is a very concrete proposal for the answer!

$$k \geq N_f/2$$

To see that this is a plausible guess, recall that the phase transition in the phase diagram above with $k \gg 1$ has a dual description through Boson-Fermion duality in terms of

$$U\left(k+\frac{N_f}{2}\right)$$

gauge theory coupled to N_f fundamental bosons [...Aharony et al., Minwalla et al....] and with Chern-Simons level $-N$.

For one sign of the boson mass we just integrate it out and get $U\left(k+\frac{N_f}{2}\right)_{-N}$ TQFT which by level-rank duality is the same as $SU(N)_{k+\frac{N_f}{2}}$, matching onto the positive mass phase of the fermionic theory.
For the other sign, the bosons condense and Higgs the gauge symmetry to $U\left(k-\frac{N_f}{2}\right)$ which therefore leads to the infrared TQFT $U\left(k-\frac{N_f}{2}\right)_{-N}$, which is level rank dual to $SU(N)_{k-\frac{N_f}{2}}$ matching the negative mass phase of the fermionic theory.

This duality makes no sense for $k < N_f/2$ because the Higgs phase is gapless and one therefore does not reproduce the correct asymptotic phases. So one is compelled to identify

$$k \geq N_f/2$$

as the regime where the phase diagram is the simple one above.

The key question is what happens for $0 \leq k < N_f/2$. Since the theory is strongly coupled in this regime (for small masses) there is some guess-work involved. Luckily we found a possible scenario that passes all the consistency checks and satisfies all the anomalies we could study.

First we start from $m = k = 0$ (which is a time reversal invariant theory) and we make the following conjecture:

$SU(N) + N_f\ \psi$, $m = k = 0 \longrightarrow\longrightarrow \mathcal{M} = \frac{U(N_f)}{U\left(\frac{N_f}{2}\right)\times U\left(\frac{N_f}{2}\right)}$ and the sigma model is accompanied by a Wess-Zumino term, Γ, whose coefficient is N.

In short, the symmetry is broken as

$$U(N_f) \to U\left(\frac{N_f}{2}\right) \times U\left(\frac{N_f}{2}\right)$$

The UV theory has time reversal symmetry anomaly $\nu_{UV} = 2N_f N \mod 16$. It is not trivial to compute the time reversal anomaly of the infrared sigma model. The Wess-Zumino term Γ is crucial in order to get the right answer.

In the particular case of $N_f = 2$, this is the $\mathbb{C}P^1$ model at $\theta = \pi$ discussed in [Freed,ZK,Seiberg]. In work in progress [Hason,ZK,Thorngren] we show that the anomaly in fact matches. In addition, we show that all the discrete flavour symmetry anomalies match.

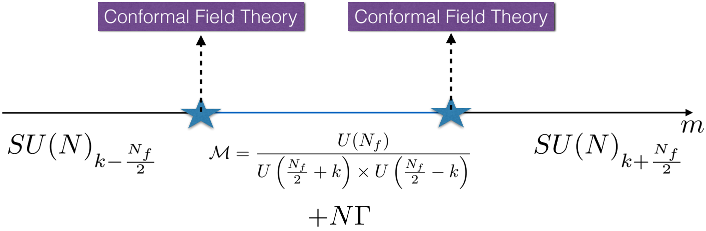
In general, we propose the following symmetry breaking pattern

$$U(N_f) \to U\left(\frac{N_f}{2}+k\right) \times U\left(\frac{N_f}{2}-k\right)$$

(and the sigma model again needs to be accompanied by a Wess-Zumino term).

This makes sense only for $k < N_f/2$ which is another way to understand the boundary between this phase and the "large k" phase.

$$SU(N) + N_f\psi, \quad \text{level } k \qquad\qquad 0 \le k < \frac{N_f}{2}$$



As in the previous case, the transition has a new dual description. For instance, for the left transition it is given by

$$U\left(\frac{N_f}{2} - k\right)_{-N} + N_f \tilde{\phi} \ ,$$

and the right transition by

$$U\left(\frac{N_f}{2} + k\right)_{-N} + N_f \phi \ .$$

While the quantum phases are robust, the transitions in these various cases may or may not be 2nd order.

See [...Jensen-Karch, Armoni-Niarchos, Argurio-Bertolini-Bigazzi-Cotrone-Niro...] for stringy constructions of some of these dualities and symmetry breaking phases. See also [Karthik-Narayanan] for a recent lattice study, (strongly) hinting that symmetry breaking indeed takes place!

We will now mention (very) briefly a striking parallel between the problem of $3d$ dynamics that we have been analyzing so far and the problem of $4d$ dynamics and domain walls. I will first explain basic idea of how the familiar four dimensional quarks get **"magnetized"** under some circumstances.

Consider $SU(N)$ Yang-Mills Theory

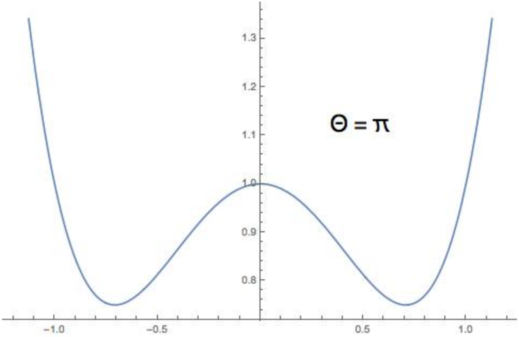
$$\mathcal{L} = \frac{-1}{4g^2} Tr(F_{\mu\nu}^2) + \frac{\theta}{8\pi^2} \star Tr(F \wedge F) \ .$$

The parameter θ is classically invisible. But quantum mechanically it matters mod 2π, $\theta \simeq \theta + 2\pi$.

Symmetries:

- 1. One-form symmetry associated to the center of the gauge group, which does not act on the dynamical fields (but acts on Wilson lines): $\mathbb{Z}_N$.
- 2. Time-reversal symmetry at $\theta = 0, \pi$.

It turns out that there is an anomaly at $\theta = \pi$! Therefore, the ground state cannot be trivial. The simplest way to saturate the anomaly is to assume that time-reversal symmetry is spontaneously broken at $\theta = \pi$. In other words, there is a first order transition as we change θ.



Let us consider a domain wall between the two degenerate vacua at $\theta = \pi$. Due to some discrete anomaly inflow, the domain wall cannot be trivial! In particular, probe quarks cannot be confined on the wall. It turns out that

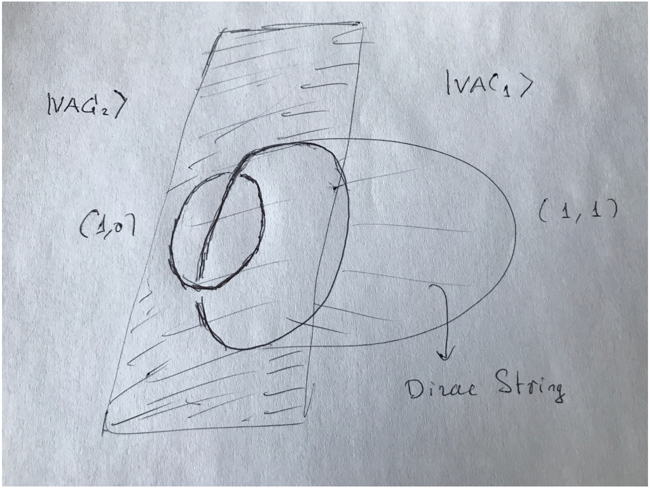
$$SU(N)_1$$

Chern-Simons theory satisfies the anomalies.

Imagine a probe quark in the bulk. It is confined and attached to a string. But when we get close to the domain wall, the quarks get deconfined!

In addition, the quarks acquire flux attachment that makes them into anyons. Even though the original spin of the quarks is $1/2$, they behave as anyons close to the domain wall. For example, the spin of the fundamental quark is now $\frac{1}{2N} + \frac{1}{2} \mod 1$.

One can think about it as due to the fact that on one side of the wall we have a condensation of a dyon and on the other side of the wall a condensation of a monopole.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Dirac String

We summarize by saying that we can loosely describe the theory on the wall as a theory of deconfined anyonic quarks. We can write a 2+1 dimensional Lagrangian with the same symmetries and anomalies

$$\mathcal{L} = \frac{-1}{4g_{3d}^2} Tr(F_{\mu\nu}^2) + \frac{1}{4\pi} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \ .$$

We expect that $g_{3d}^2 \sim \Lambda_{YM}$.

We could say that there is topological order on the wall. Now, what if the quarks become dynamical? Well, for as long as they are sufficiently heavy, nothing happens, since the topological order cannot disappear. So we still have $SU(N)_1$ on the wall. But now let us take them to be very light.

First, the Lagrangian we are studying is $\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{quark}$

$$\mathcal{L}_{gauge} = \frac{-1}{4g^2} Tr(F_{\mu\nu}^2) + \frac{\theta}{8\pi^2} Tr(F \wedge F)$$

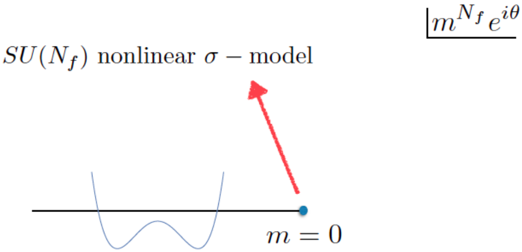
$$\mathcal{L}_{quark} = i \sum_I \bar{\psi}^I \not{D} \psi_I + i \sum_I \bar{\tilde{\psi}}_I \not{D} \tilde{\psi}^I + m \sum_I \psi_I \tilde{\psi}^I + c.c.$$

where m real and positive without loss of generality and $\theta = \pi$.

$m \gg \Lambda,\ \theta = \pi$: As we said, we integrate out the quarks and obtain pure Yang-Mills theory at $\theta = \pi$. There are two ground states and a first order phase transition as we change θ. The domain wall theory is $SU(N)_1$ TFT.

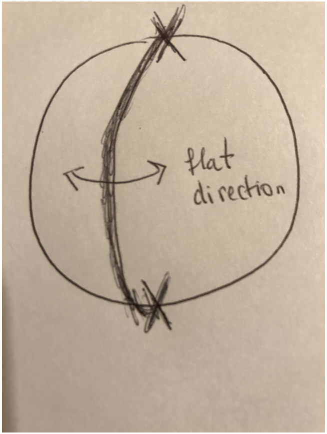
The line of first order transition must end. It ends at $m = 0$ where we have the chiral Lagrangian,

$$\mathcal{L} \sim f_\pi^2 Tr\left(\partial U \partial U^\dagger\right) \ , \qquad U \in SU(N_f) \ .$$



While the domain wall theory at large mass is $SU(N)_1$, at low mass we use the chiral Lagrangian. At $\theta = \pi$ it has two degenerate ground states related by time reversal [Dashen...] ! furthermore it turns out to have massless degrees of freedom on the domain wall

$$\frac{SU(N_f)}{S[U(N_f-1)\times U(1)]} = \mathbb{CP}^{N_f-1} .$$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direction

The domain wall therefore undergoes a transition from $SU(N)_1$ to a massless $\mathbb{CP}^{N_f-1}$ sigma model. The exact same dynamics occurs, according to our conjecture, in

$$SU(N)_{1-N_f/2} + N_f \ \Psi \ ,$$

which is, naively, the 2+1d theory on the wall. The bosonic dual theory we presented is $U(1)_N + N_f\phi$. It is tempting to note the similarity of this to what might occur as the worldvolume theory on a D-brane in string theory.

For additional discussions of such ideas, see, for instance, the recent papers [**Benini** et al., **Cordova** et al., **Tanizaki**, **Sulejmanpasic**, **Kikuchi**, **Misumi**, **Sakai**, **Shimizu**, **Anber-Poppitz**, **Aitken-Cherman-Ünsal**, **Yao-Hsieh-Oshikawa**, **Dunne**, **Hofman-Iqbal**,**Guo-Putrov-Wang**, **Yamazaki-Yonekura**, **Kitano** et al., **Di Vecchia** et al., **Dierigl-Pritzel**, **Draper**, **Ritz-Shukla**, **Armoni-Niarchos**, **Argurio-Bertolini-Bigazzi-Cotrone-Niro** (embedding in the Sakai-Sugimoto model – The bosonic dual appears naturally!), **Aitken-Baumgartner-Karch**].

- We have presented some very concrete predictions for the dynamics of 2+1 dimensional QCD. Topological order plays an important role. We proposed a new symmetry breaking pattern and new dualities. These proposals pass a large amount of nontrivial consistency checks.
- We have discussed a natural embedding of the dynamics in four-dimensional gauge theories through domain walls. We argued that the quarks acquire flux attachment in this way. We have shown that some of the phase transitions, symmetry breaking patterns, and dualities that we proposed in 3d appear naturally from four dimensional dynamics.