HOLOGRAPHIC ENTANGLEMENT

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March 16, 2018

BAY AREA PARTICLE THEORY SEMINAR

based on earlier works w/ {M. Headrick, A. Lawrence, H. Maxfield, M. Rangamani, T. Takayanagi, E. Tonni}
& on work in progress w/ M. Headrick

OUTLINE

- Context
  - Background: AdS/CFT
  - Motivation: covariantize to elucidate the correspondence
- Holographic Entanglement
  - Entanglement entropy
  - RT & HRT
- Recasting holographic entanglement
  - Bit threads
  - Covariant bit threads
- Summary

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Gauge/gravity duality

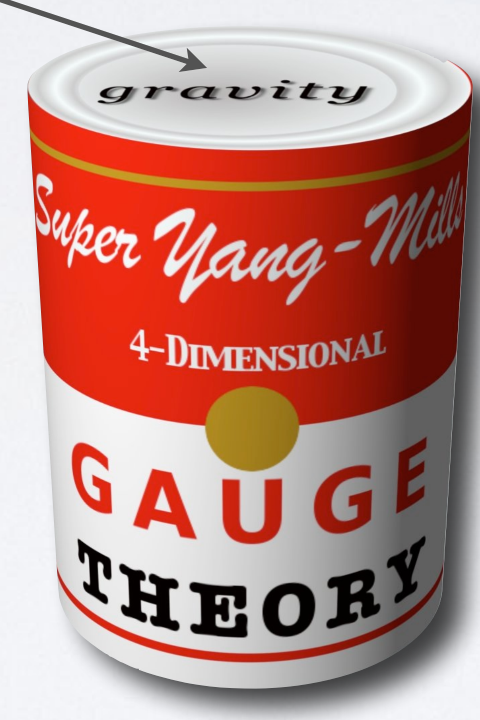
also known as the AdS/CFT correspondence

String theory (gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS

"on boundary"

'soup can' diagram of AdS:

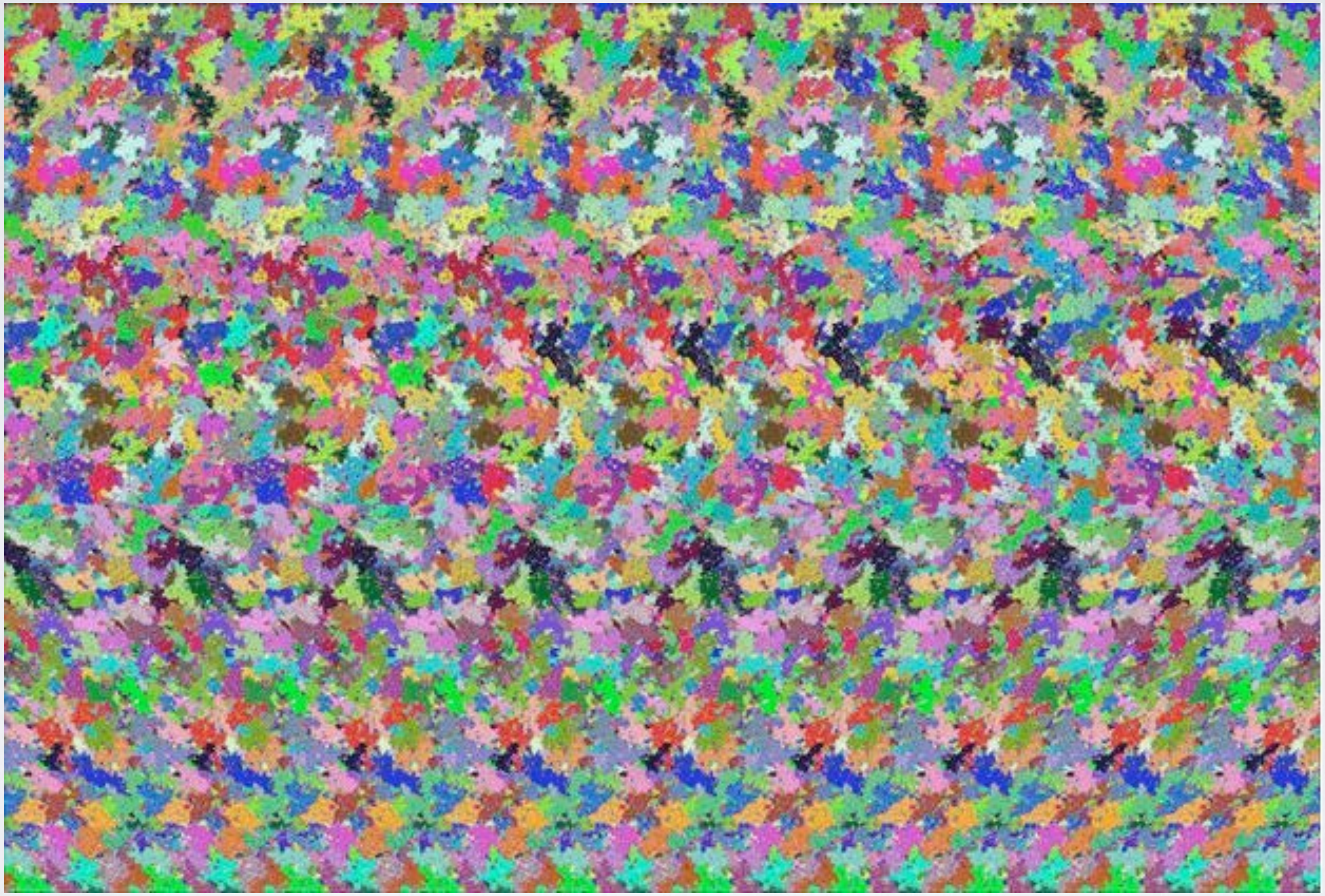


here label is everything...

- Since the two descriptions live in different number of dimensions, we call such a correspondence holographic.

Gauge/gravity duality

* better analogy: stereogram...

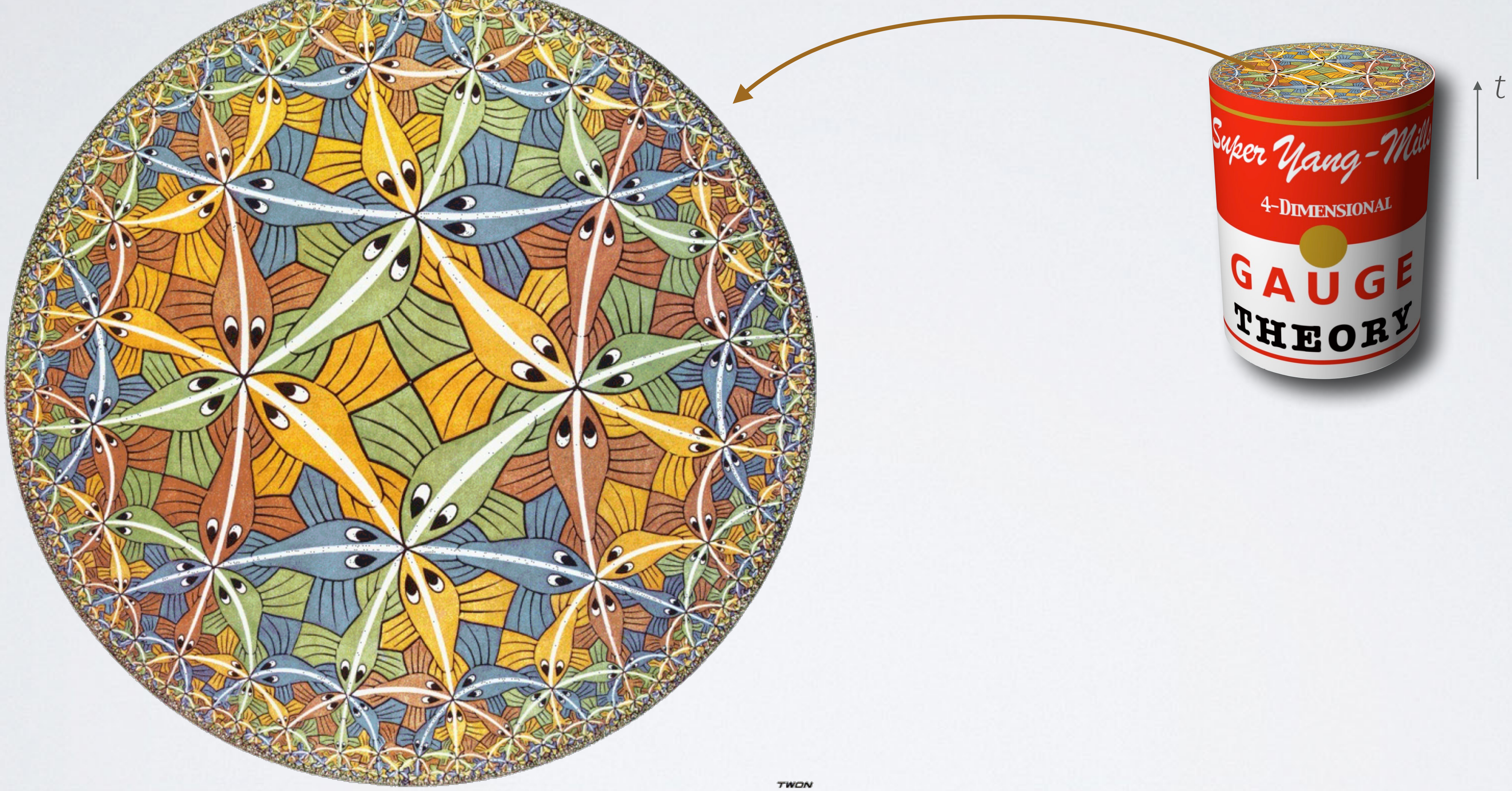


...but infinitely more complicated

Radial direction

What boundary quantity encodes the extra bulk direction?

- hints from spatial geometry of AdS:



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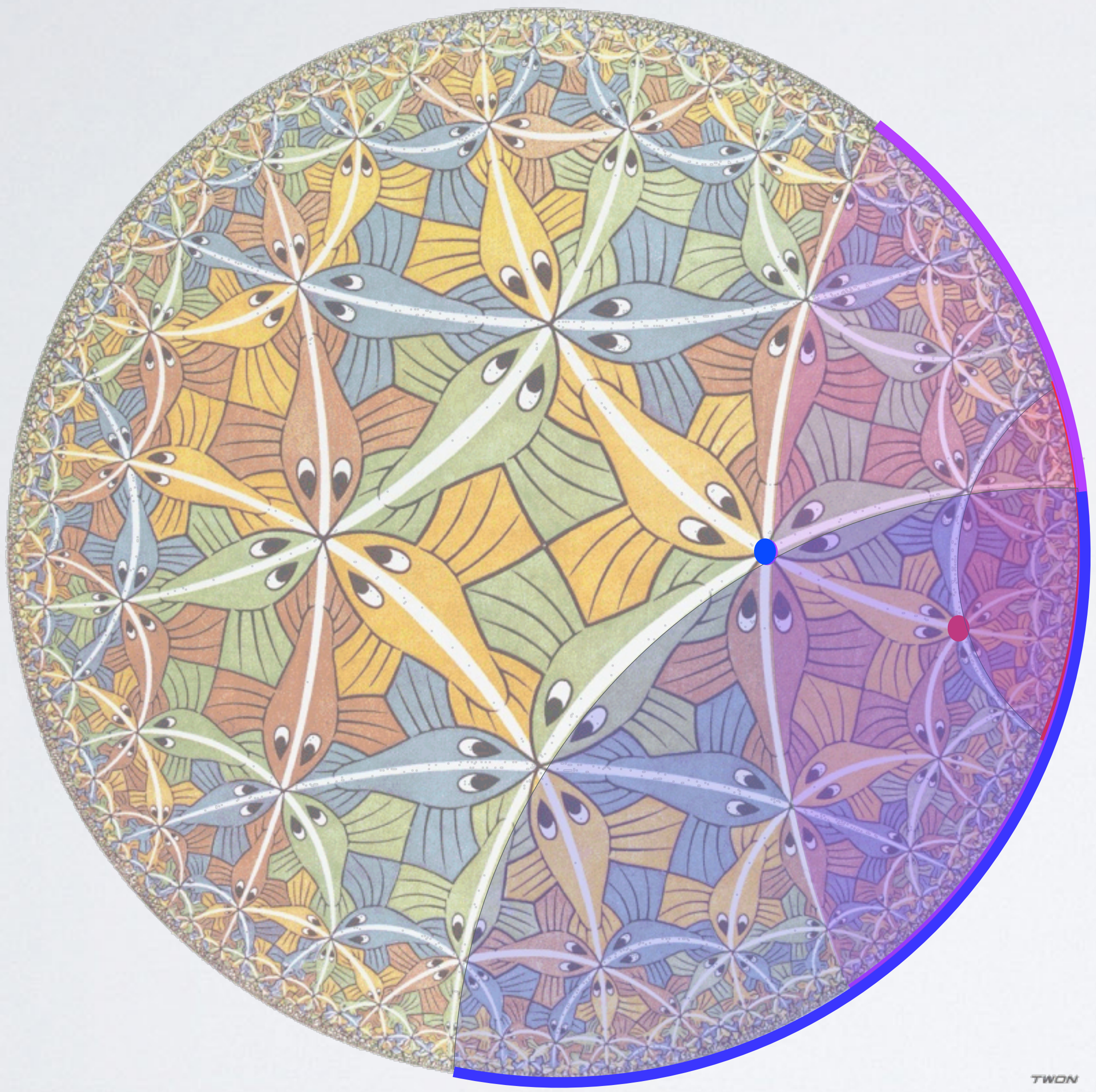


The radial bulk direction comes from a scale size on boundary:

- points near the boundary ⟷ small arcs
- points further in the bulk ⟷ larger arcs
- same point represented by multiple arcs

- Provides useful intuition:

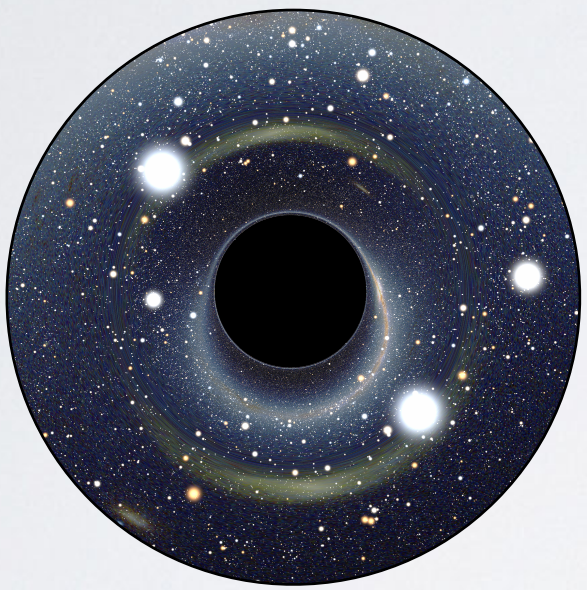
e.g. bulk particle falling due to gravity ⟷ boundary excitation spreading outward
… falling into black hole ⟷ … thermalizing

Applications of gauge/gravity duality

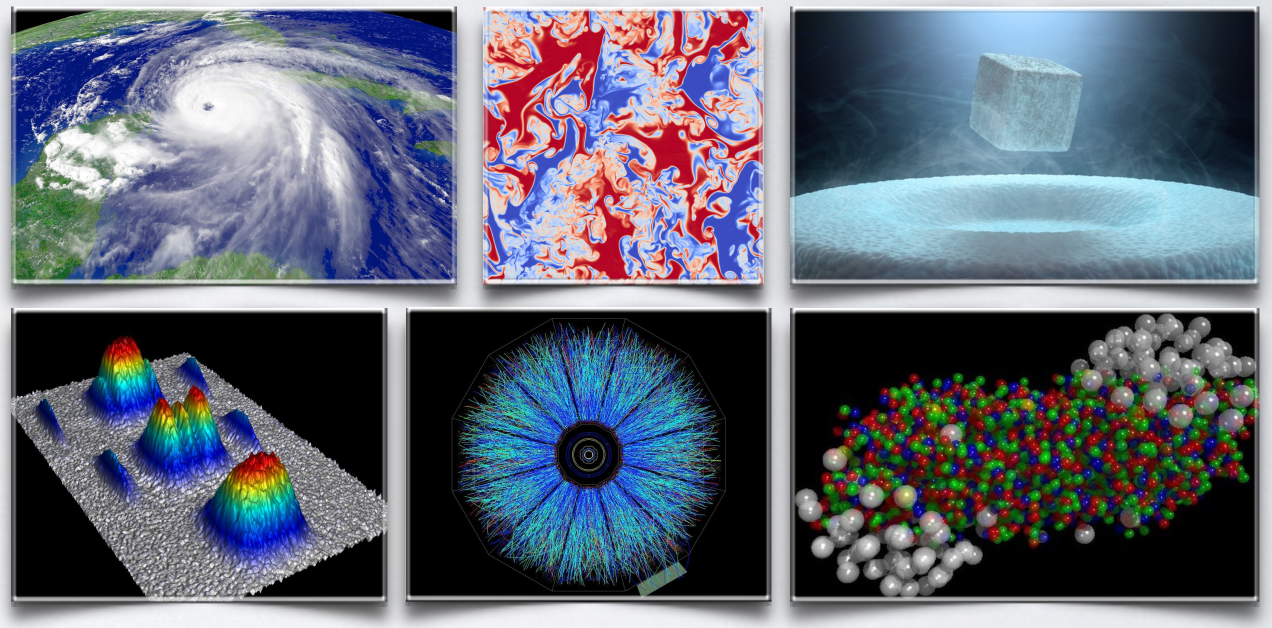
String theory (gravity) $\Longleftrightarrow$ field theory (no gravity)

"in bulk" = higher dimensions — *"on boundary" = lower dimensions*

describes gravitating systems, e.g. black holes



describes experimentally accessible systems



Invaluable tool to:

- Study strongly interacting field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?
- (How) does the CFT "see" inside a black hole?
  - Does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Main message: using GR technology goes a long way…

Motivation

- Elucidate holography
  - Fundamental nature of spacetime & its relation to entanglement
  - Structure/characterization of CFTs (& states) w/ gravity dual
- Start w/ situations with large amount of symmetry (e.g. pure AdS)
  - Explicit calculations possible, can obtain analytical expressions
  - Use these to guess duality relations → entry in gauge/gravity dictionary
- But this has limitations
  - How to generalize? (e.g. time dependence)
  - Often symmetry brings degeneracy between logically distinct concepts
- Need to "covariantize"
  - Define a quantity which is purely geometrical (e.g. independent of any choice of coordinate systems) and fully general

Utility of covariant constructs

- Gives a general prescription
  - Definition of a quantity is equally robust on both sides of duality
  - Once beyond analytically tractable cases, might as well go for full generality (within the class of systems we want to consider, i.e. $N = \infty$)
- Time dependence interesting in its own right
  - Novel phenomena in out-of-equilibrium systems
  - New insight into the structure of the theory
- Breaks degeneracy between distinct constructs
  - Allows us to identify the true dual → underlying nature of the map
- Natural covariant constructs motivate new relations
  - Even if a given construct is not the sought dual, it eventually finds its use

Example: Holographic Entanglement Entropy

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Paths to Holographic Entanglement

String theory ($\ni$ gravity) $\Longleftrightarrow$ gauge theory (CFT)

"in bulk" asymp. AdS $\times$ K

"on boundary"

Applied AdS/CFT:

- study specific system via its dual
- e.g. AdS/QCD, AdS/CMT, …

Fundamentals of AdS/CFT:

- why/how does the duality work
- map between the 2 sides

Holographic Entanglement Entropy

Quantum Gravity

Entanglement

- Most non-classical manifestation of quantum mechanics
  - "Best possible knowledge of a whole does not include best possible knowledge of its parts — and this is what keeps coming back to haunt us" [Schrodinger '35]
- New quantum resource for tasks which cannot be performed using classical resources [Bennet '98]
- Plays a central role in wide-ranging fields
  - quantum information (e.g. cryptography, teleportation, …)
  - quantum many body systems
  - quantum field theory
- Hints at profound connections to geometry…

Entanglement Entropy (EE)

Suppose we only have access to a subsystem A of the full system = A + B. The amount of entanglement is characterized by Entanglement Entropy S_A :

- reduced density matrix $\rho_A = \mathrm{Tr}_B \, |\psi\rangle\langle\psi|$

  (more generally, for a mixed total state, $\rho_A = \mathrm{Tr}_B \rho$)

- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \, \log \rho_A$

Defined if we can divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

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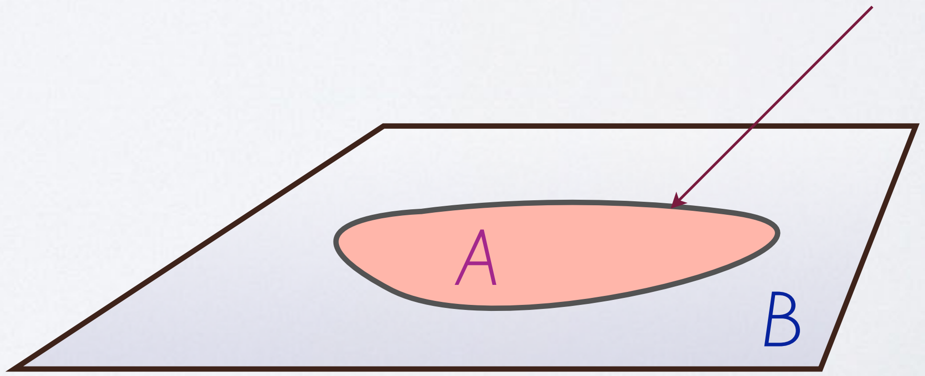
- e.g. in local QFT:

  A and B can be spatial regions, separated by a smooth entangling surface



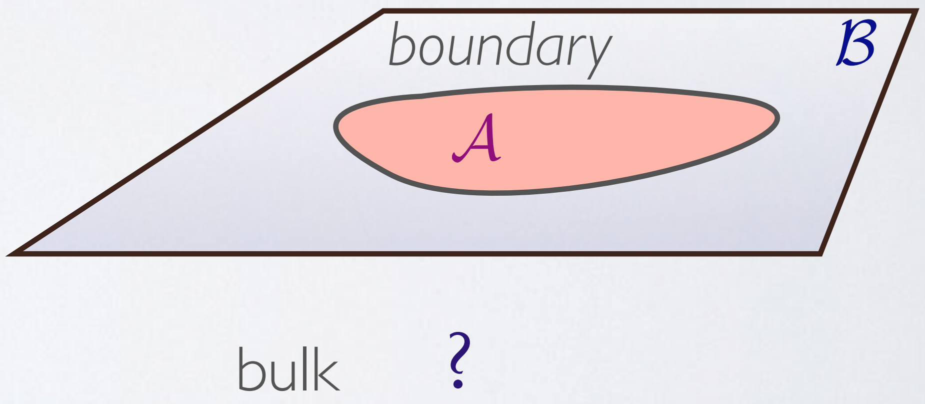
The good news & the bad news

- But EE is hard to deal with…
  - non-local quantity, intricate & sensitive to environment
  - difficult to measure
  - difficult to calculate

… especially in strongly-coupled quantum systems

- AdS/CFT to the rescue?

~ Is there a natural bulk dual of EE?
(= "Holographic EE")

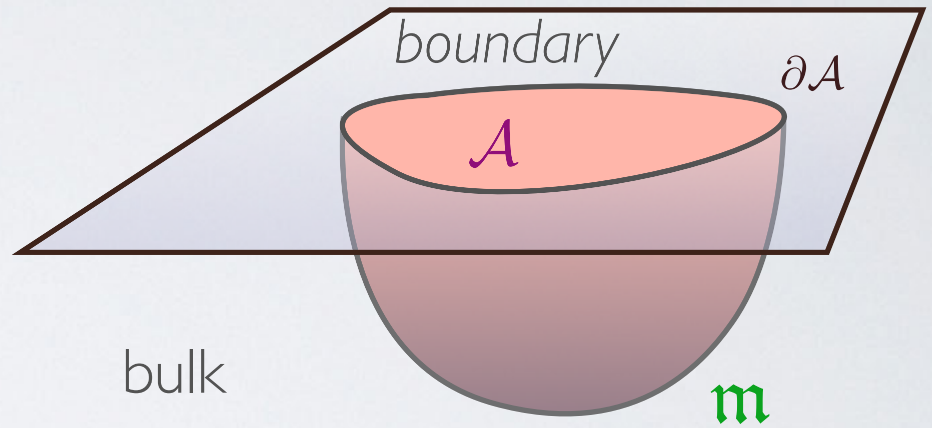


Yes! - described geometrically…

Holographic EE in static situations

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* anchored on entangling surface $\partial\mathcal{A}$ & homologous to $\mathcal{A}$



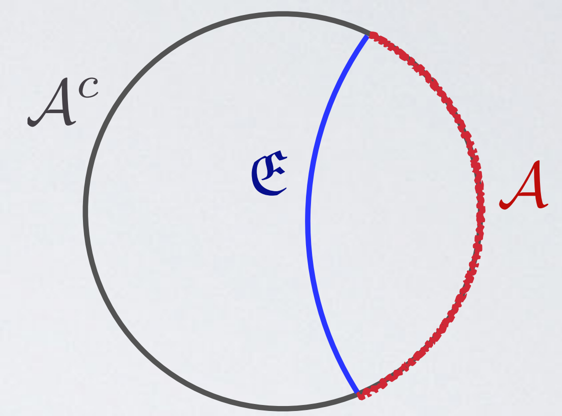
$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$

Remarks:

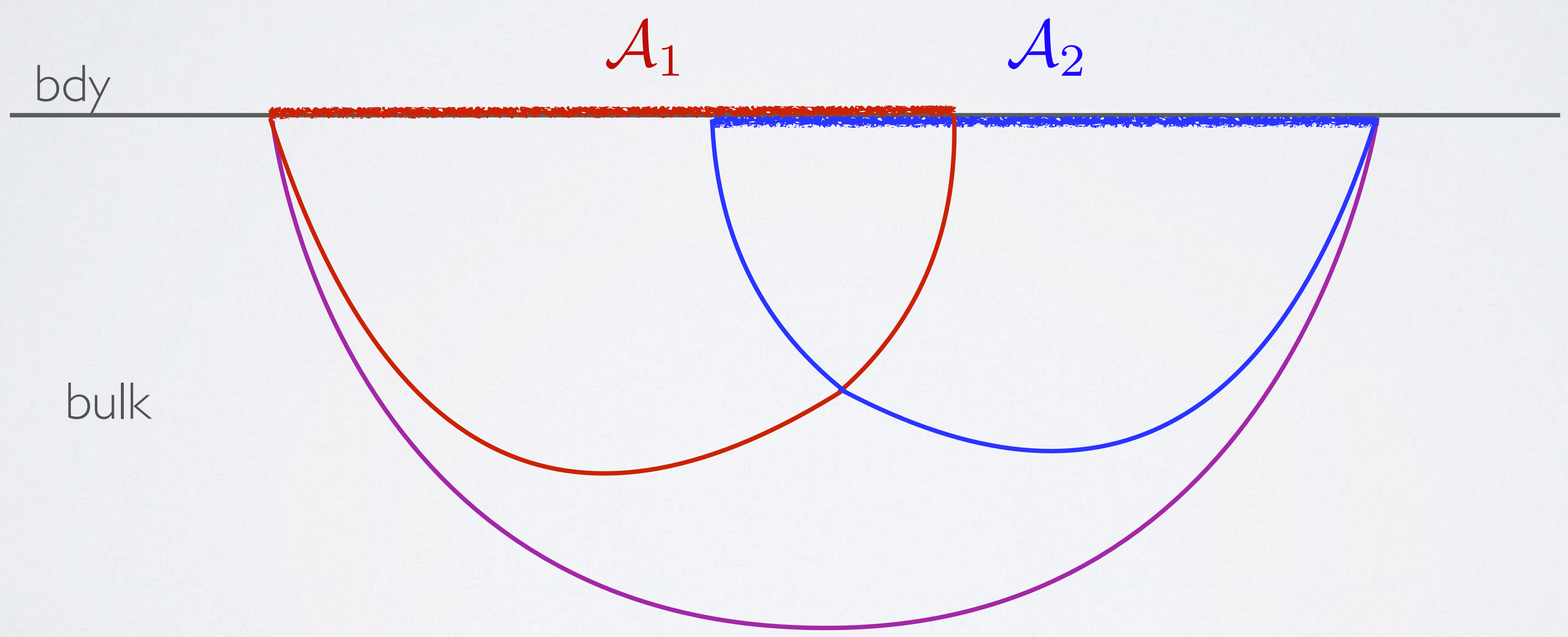
- Large body of evidence, culminating in [Lewkowycz & Maldacena '13]
- cf. black hole entropy…
- Minimal surface "hangs" into the bulk due to large distances near bdy.
- Note that both LHS and RHS are in fact infinite

Manifest properties of EE

- For pure states $S_{\mathcal{A}} = S_{\mathcal{A}^c}$
- Positivity: $S_{\mathcal{A}} \geq 0$



- Subadditivity: $S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2}$



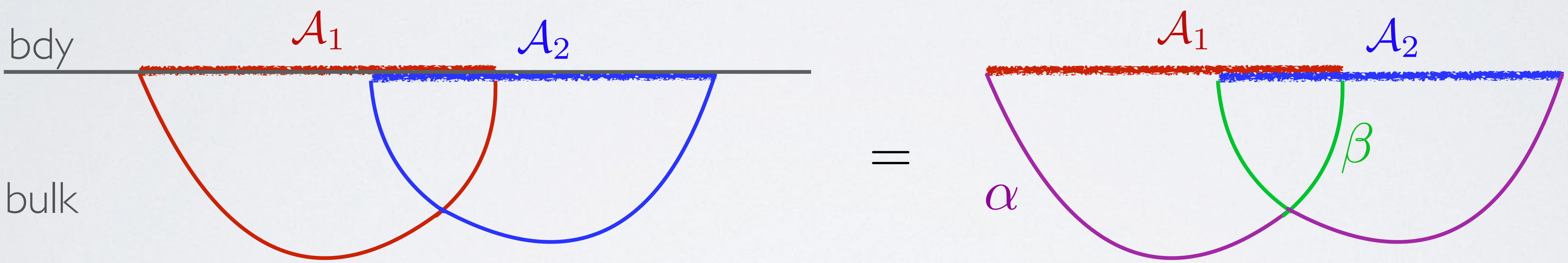
- Implies positivity of mutual information: $I(\mathcal{A}_1, \mathcal{A}_2) = S_{\mathcal{A}_1} + S_{\mathcal{A}_2} - S_{\mathcal{A}_1 \cup \mathcal{A}_2}$

Proof of Strong Subadditivity

- strong subadditivity:

$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

- proof in static configurations [Headrick & Takayanagi]



$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta$$

Proof of Strong Subadditivity

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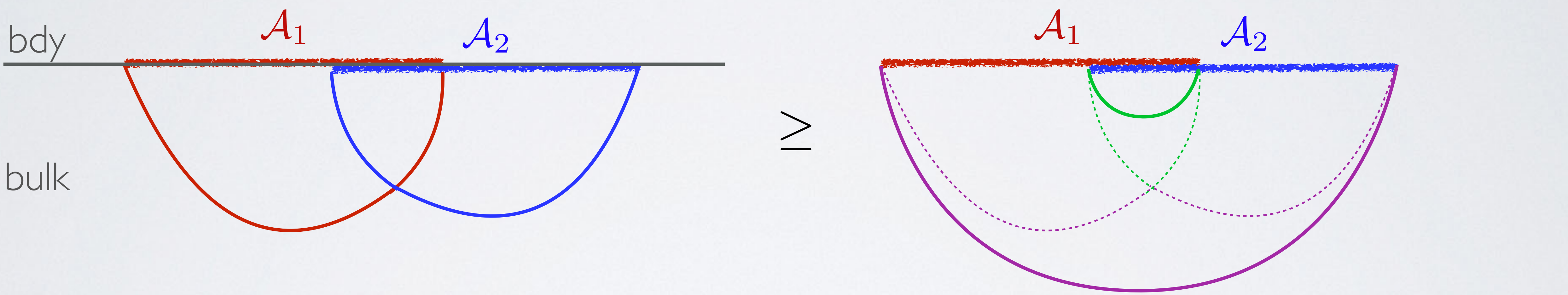


$$S_{\mathcal{A}_1} + S_{\mathcal{A}_2} = \alpha + \beta \geq S_{\mathcal{A}_1 \cup \mathcal{A}_2} + S_{\mathcal{A}_1 \cap \mathcal{A}_2}$$

- Similarly prove monogamy of mutual information [Hayden, Headrick, Maloney] valid in holography but not in general: $S_A + S_B + S_C + S_{ABC} \leq S_{AB} + S_{BC} + S_{AC}$

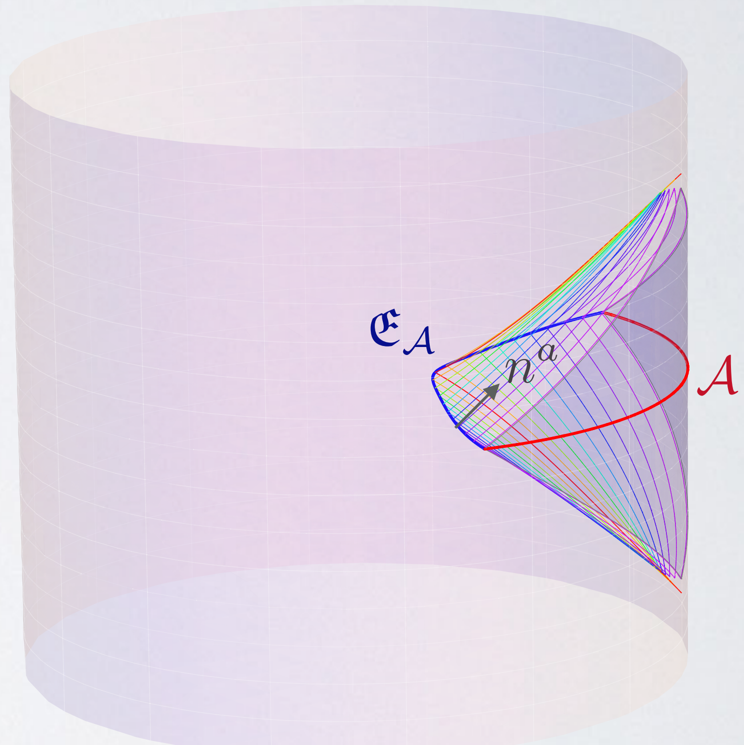
Bulk dynamics from EE?

- We can in principle decode the bulk geometry from $\{S_{\mathcal{A}}\}$ for a suitable set of $\mathcal{A}$'s.
- But can we extract bulk dynamics more directly?
  - Use the strong subadditivity property of EE:

$$\delta^2_{\mathcal{A}} \, S_{\mathcal{A}} \sim \int_{\mathfrak{E}_{\mathcal{A}}} E_{ab} \, n^a \, n^b \geq 0$$

specific 2nd order variation of region

cf. Null Energy Condition



  - proved at linearized level in 3-d, but conjectured to hold more generally…

[Bhattacharya, VH, Rangamani, Takayanagi, '14]
cf. [Lashkari, Rabideau, Sabella-Garnier, Van Raamsdonk]

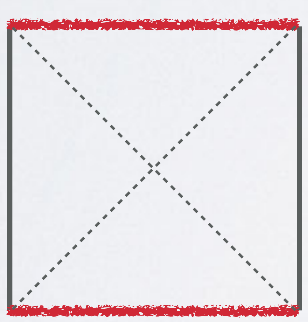
Spacetime from entanglement?

How does bulk spacetime emerge in the first place?

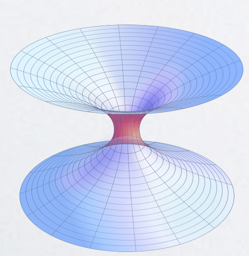
- Some connected spacetimes emerge as superpositions of disconnected spacetimes [Van Raamsdonk; Swingle]

eg. eternal AdS black hole as thermofield double:

$$|\psi\rangle = \sum_i e^{-\frac{\beta E_i}{2}} |E_i\rangle \otimes |E_i\rangle$$



- Entanglement builds bridges: 'ER = EPR' [Maldacena, Susskind]

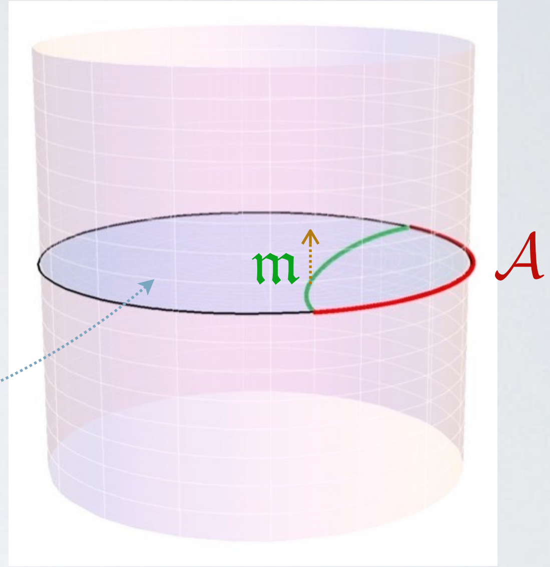


Einstein-Rosen bridge ↙ ↘ Einstein-Podolsky-Rosen entanglement

Covariant Holographic EE

But the RT prescription is not well-defined outside the context of static configurations:

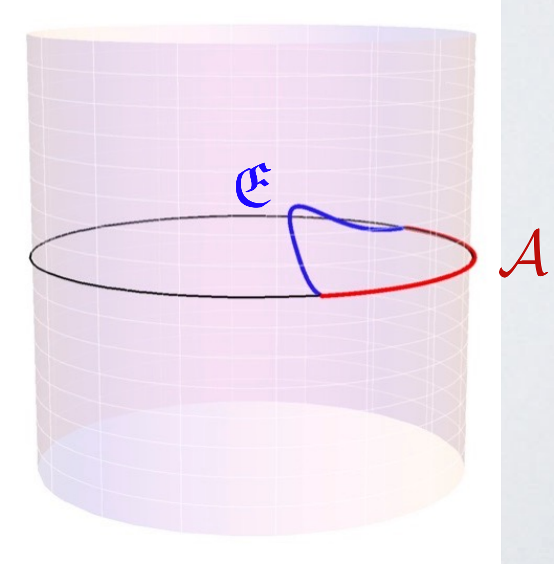
- In Lorentzian geometry, we can decrease the area arbitrarily by timelike deformations
- In time-dependent context, no natural notion of "const. t" slice…



In *time-dependent* situations, RT prescription must be covariantized:

Simplest candidate: [HRT = VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk



This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime
⇒ equally robust as in CFT

"Pf" in [Dong, Lewkowycz, Rangamani '16]

Curious features of EE:

- Extremal surfaces can have intricate behavior:
  - $\mathfrak{E}$ can have discontinuous jumps under smooth variations of $\mathcal{A}$

    $\rightarrow$ phase transitions in EE
  - $\mathfrak{E}$ can be topologically nontrivial even for simply-connected regions $\mathcal{A}$
- Holographic EE seems too local:
  - sharply-specified both on boundary and in bulk
  - but: $\rightarrow$ we can reconstruct the bulk metric (modulo caveats) solely from the set $\{S_{\mathcal{A}}\}$ for a suitable set of $\{\mathcal{A}\}$
- Holographic EE seems too non-local:
  - global minimization condition + homology constraint makes $S_{\mathcal{A}}$ sensitive to arbitrarily distant regions in the bulk…

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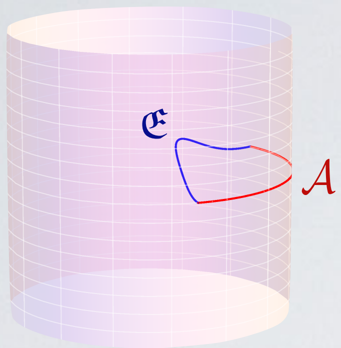
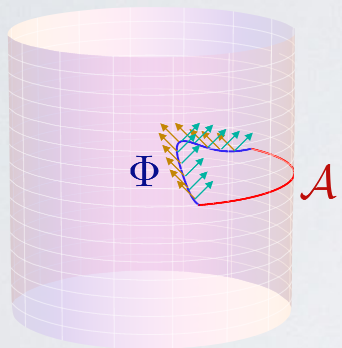
Covariant re-formulations

- Covariance is pre-requisite to construct being physically meaningful, but it need not be unique
  - Distinct geometrical formulations can turn out equivalent
- This redundancy is useful
  - Each formulation can have its own advantages
  - e.g. different properties may be manifest in different formulations (cf. gauge / coordinate choice)
  - Re-formulation can reveal deeper relations (cf. ER=EPR [Maldacena, Susskind])

Covariant re-formulations of HEE



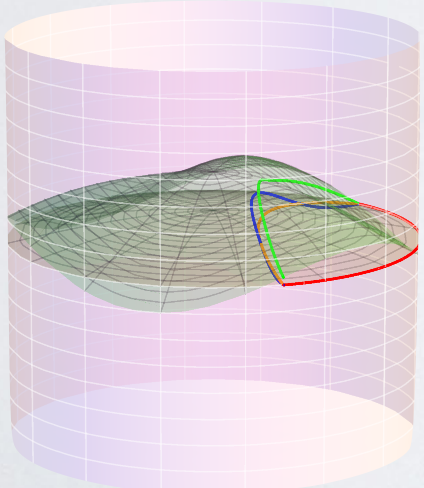
- $\mathfrak{E}$ = Extremal surface
  - (relatively) easy to find
  - minimal set of ingredients required in specification
  - need to include homology constraint as extra requirement



- Φ = Surface with zero null expansions
  - cf. light sheet construction & covariant entropy bound [Bousso, '99]:

    Bulk entropy through light sheet of surface $\sigma \leq$ Area(σ)/4

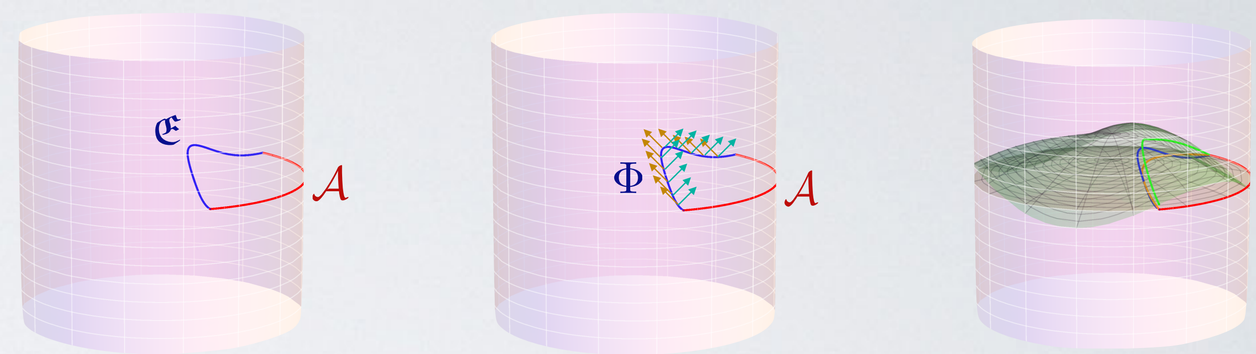
    Φ = surface admitting a light sheet closest to bdy



- Maximin surface [Wall, '12]
  - maximize over minimal-area surface on a spacelike slice
  - requires the entire collection of slices & surfaces
  - implements homology constraint automatically
  - useful for proofs (e.g. SSA)

Covariant re-formulations of HEE

All of these are the same geometrical construct.



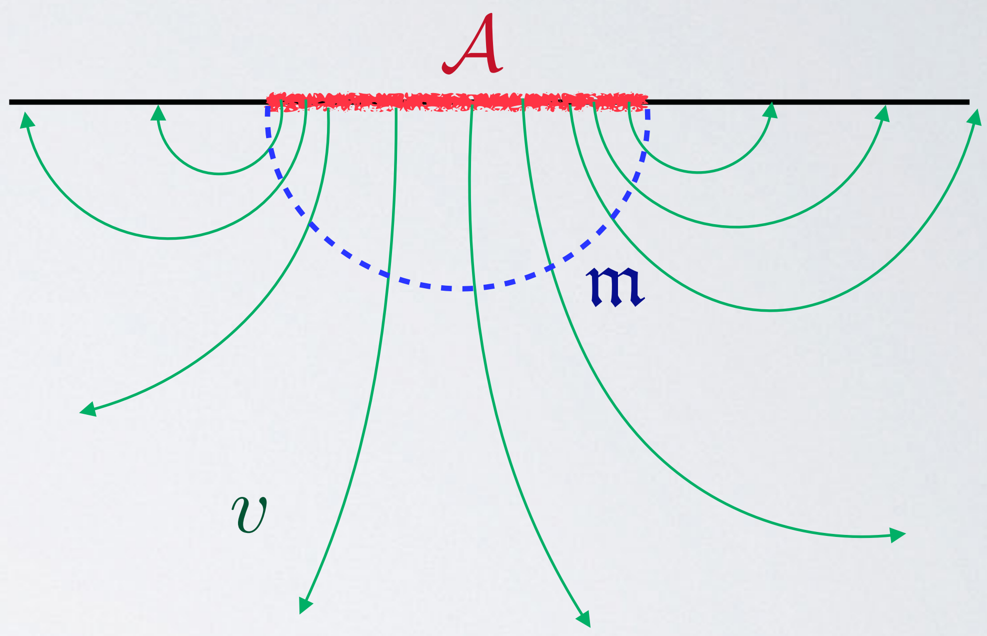
BUT it does not elucidate the relation to quantum information:

- Where does the information live?
- Mutual information I(A:B) = S(A) + S(B) - S(AB) is given by surfaces located in different spacetime regions.
- Geometric proof of SSA (S(AB) + S(BC) ≥ S(B) + S(ABC)) obscures its meaning as monotonicity under inclusion of correlations

Bit thread picture of (static) EE

- Reformulate EE in terms of flux of flow lines [Freedman & Headrick, '16]
  - let v be a vector field satisfying $\nabla \cdot v = 0$ and $|v| \leq 1$. Then EE is given by

$$S_{\mathcal{A}} = \max_{v} \int_{\mathcal{A}} v$$

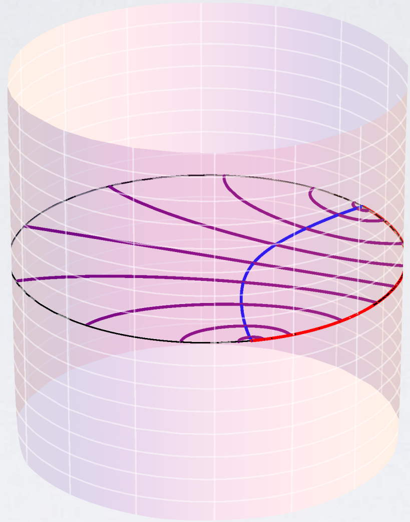


  - By Max Flow - Min Cut theorem, equivalent to RT: (bottleneck for flow = minimal surface)

- Useful reformulation of holographic EE
  - behaves more naturally
  - is more computationally efficient
  - ties better to QI quantities
  - provides more intuition

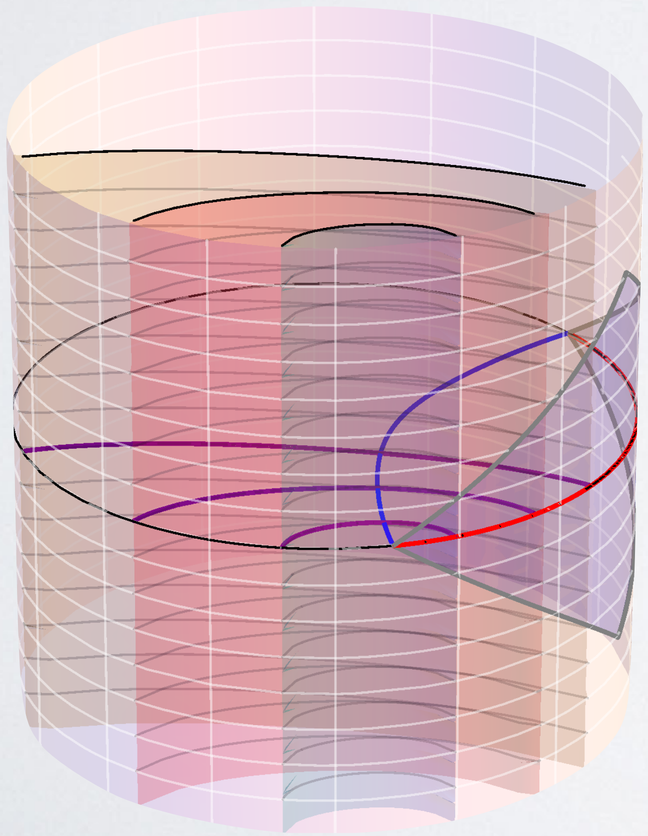
- How does this extend to time-dependent settings?

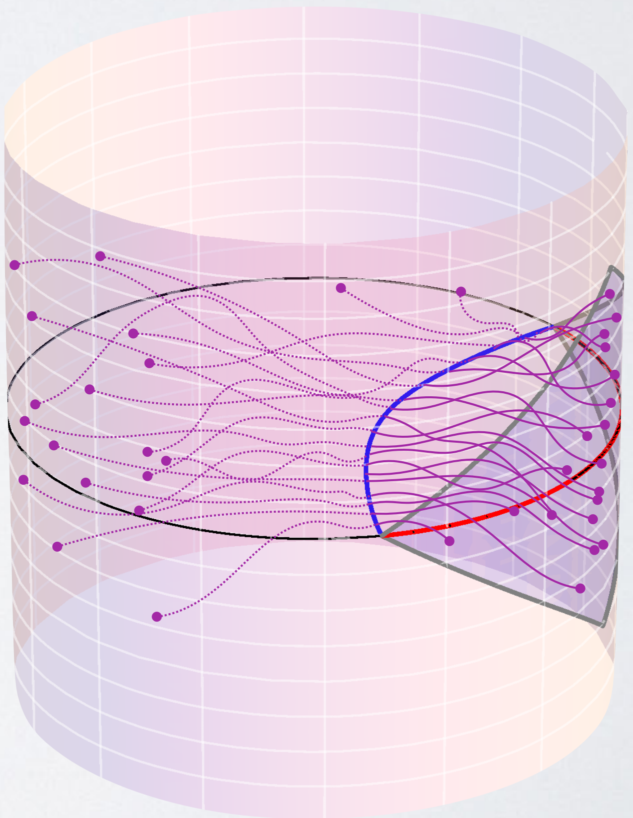
Natural possibilities



extend threads in time
flow sheets

keep 1-d threads
flow lines





Convex optimization as a tool

- Max-flow/min-cut is an example of Lagrangian duality in theory of convex optimization

- Setup:
  - Convex program P: minimize $f_0(y)$ over $y \in \mathcal{D}$ such that $\forall i,\ f_i(y) \leq 0,\ \forall j,\ h_j(y) = 0$

    convex domain

    convex functions

    affine functions

  - More general problems may be converted to the requisite form via convex relaxation

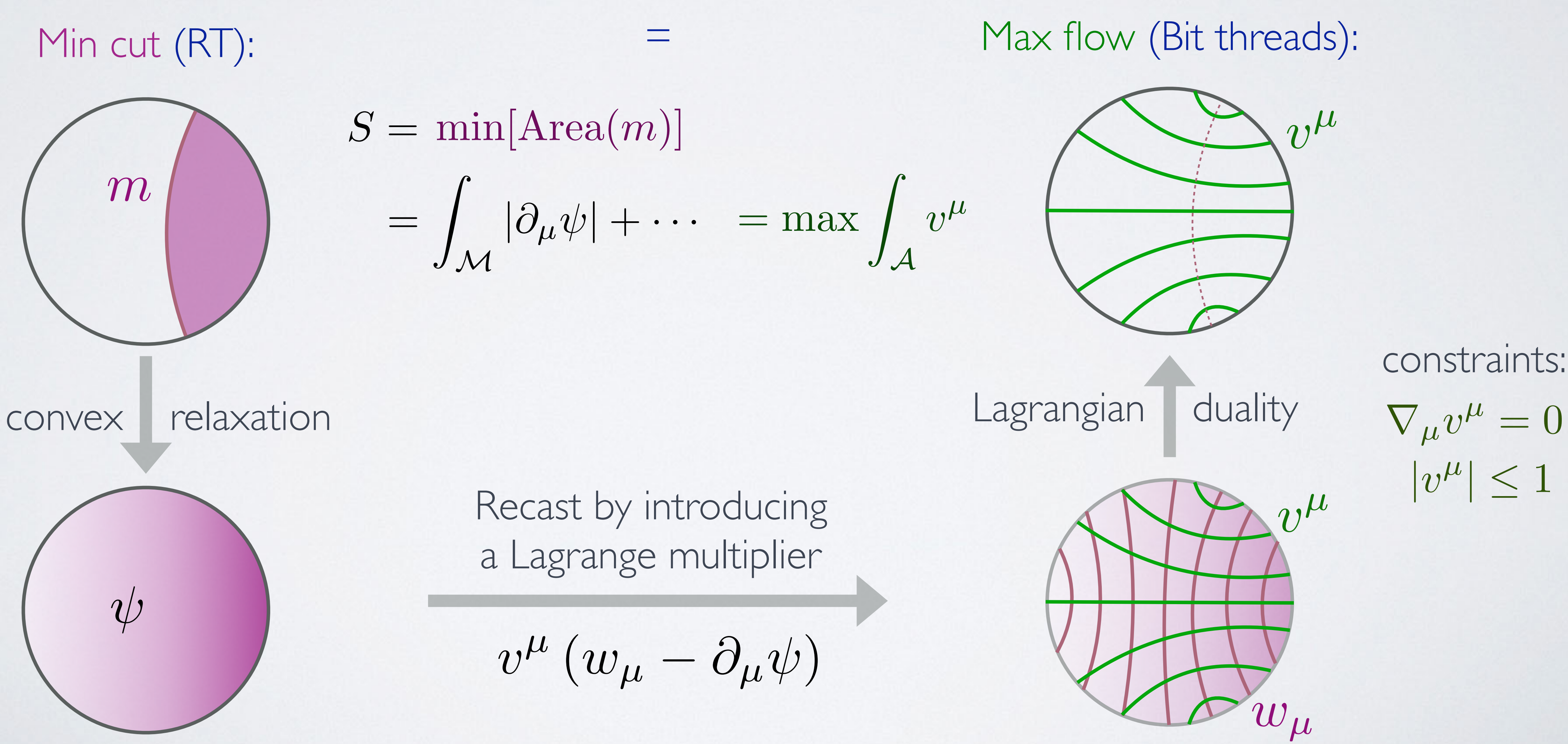
  - use Lagrange multipliers $$L(y, \lambda, \nu) \equiv f_0(y) + \sum_i \lambda_i\, f_i(y) + \sum_j \nu_j\, h_j(y)$$

  - solution via convex optimization: $$p^* = \inf_y \sup_{\lambda, \nu} L(y, \lambda, \nu)$$

- Lagrangian duality: swap order
  - new extremization problem, in new variables

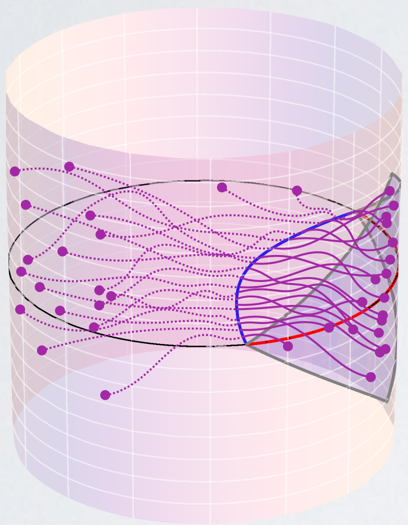
Convex optimization as a tool

- Strategy:
  - Formulate the (Lorentzian) min cut side as convex relaxation
  - Interpret the dual geometrically
- Eg. for static case:



Flow lines

- Via HRT → maximin → maximax → convex relaxation & Lagrangian duality, we still get flow lines = covariant bit threads…



But what is the QI interpretation ?

- Entanglement entropy counted by events ?
  - e.g. # of indep. measurements that can be performed within $D[\mathcal{A}]$
  - novel interpretation…
- Why are 1-d structures natural?
  - why is a specific measurement connected to another instantaneous event somewhere in $\mathcal{A}^c$?

Summary & Outlook

- Holography conveniently geometrizes entanglement
  - Finding bulk geometrical constructs is (relatively) easy!
  - Useful in proving important properties!
  - Why is EE related to geometry so simply?
  - Duals of other measures of entanglement?
- General covariance is a powerful guiding principle
  - Motivated subregion/subregion duality
  - Covariantize bit threads to elucidate essence of holographic EE
  - Significance of instantaneous nature: (Why) 1-d threads?
- Convex relaxation and Lagrangian duality is a powerful tool
  - Motivates new geometric constructs, new elegant proofs, connections…
  - Other applications?
- Relation between spacetime (gravity) and entanglement?


Thank you